Weighted inequalities and dyadic harmonic analysis

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Outline

1. Weighted Inequalities
2. Dyadic harmonic analysis on $\mathbb{R}$
3. Case study: Dyadic proof for commutator $[H, b]$
4. Sparse operators and families of dyadic cubes

Weighted inequalities

Question (Two-weights L^p-inequalities for operator T)

Is there a constant $C_p(u,v) > 0$ *such that*

$$\|Tf\|_{L^p(v)} \leq C_{T,p}(u,v)\,\|f\|_{L^p(u)} \ \textit{for all}\ \ f \in L^p(u)?$$

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The weights u, v are a.e. positive locally integrable functions on $\mathbb{R}^d$.
$f \in L^p(u)$ iff $\|f\|_{L^p(u)} := (\int |f(x)|^p u(x)\,dx)^{1/p} < \infty$.
Linear or sublinear operator $T : L^p(u) \to L^p(v)$.

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Goals

1. *Given operator T, identify and classify weights u, v for which the operator T is bounded from $L^p(u)$ to $L^p(v)$.*
2. *Understand nature of constant $C_{T,p}(u,v)$.*

We concentrate on *one-weight L^p inequalities*: $u = v = w$, for Calderón-Zygmund singular integral operators.

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- Petermichl's Haar shift operator Ш ("Sha"),
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CZ operators are bounded in $L^p(w)$, when the weight w is in the Muckenhoupt A_p-class (Coifman-Fefferman '74), same holds for commutators (Alvarez-Bagby-Kurtz-Pérez '96).

A_p weights

Definition

A weight w is in the *Muckenhoupt* A_p class if its A_p characteristic, $[w]_{A_p}$ is finite, where,

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}dx\right)^{p-1}, \quad 1 < p < \infty\,,$$

the supremum is over all cubes in $\mathbb{R}^d$ with sides parallel to the axes.

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Note that a weight $w \in A_2$ if and only if

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Example

In $\mathbb{R}$, $w(x) := |x|^\alpha$, $w \in A_p \Leftrightarrow -1 < \alpha < p - 1$.

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- [AIS, Proposition 22] They reduced the conjecture to showing that the Beurling transform T satisfies linear bounds in $L^p(w)$ for $p > 1$

$$\|T\phi\|_{L^p(w)} \leq C(p)[w]_{A_p}\|\phi\|_{L^p(w)}, \quad \forall w \in A_p.$$

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As it turns out $1 < q < 2$ and $p = q' > 2$ are the values of interest. Linear bounds for the Beurling transform and $p \geq 2$ were proved by Petermichl-Volberg '02. As a consequence the regularity at the borderline case $q = 2K/(K+1)$ was stablished.

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Theorem (Chung, P., Pérez '12)

Given linear operator T, if for all $w \in A_2$ there exists a $C_{T,d} > 0$ such that for all $f \in L^2(w)$,

$$\|Tf\|_{L^2(w)} \leq C_{T,d}[w]_{A_2}^{\alpha}\|f\|_{L^2(w)}.$$

then its commutator with $b \in BMO$ will satisfy,

$$\|[T, b]f\|_{L^2(w)} \leq C_{T,d}^{*}[w]_{A_2}^{\alpha+1}\|b\|_{BMO}\|f\|_{L^2(w)}.$$

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- Proof uses classical Coifman-Rochberg-Weiss '76 argument based on (i) Cauchy integral formula; (ii) quantitative Coifman-Fefferman result: $w \in A_2$ implies $w \in RH_q$ with $q = 1 + 1/2^{5+d}[w]_{A_2}$ and $[w]_{RH_q} \leq 2$; (iii) quantitative version: $b \in BMO$ implies $e^{\alpha b} \in A_2$ for α small enough with control on $[e^{\alpha b}]_{A_2}$.

Some generalizations

- Higher order commutators $T_b^k := [b, T_b^{k-1}]$ (powers $\alpha + k$, k). Sharp for all $k \geq 1$ and all dimensions, as examples involving the Riesz transforms show, with $\alpha = 1$. Extrapolated bounds are sharp for all $1 < p < \infty$, Chung, P. Pérez '12.

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- Mixed $A_2 - A_\infty$, Hytönen, Pérez '13 showed for T CZ

$$\|[T,b]\|_{L^2(w)} \leq C_n \|b\|_{BMO} [w]_{A_2}^{\frac{1}{2}} \left([w]_{A_\infty} + [w^{-1}]_{A_\infty}\right)^{\frac{3}{2}}$$

See also Ortiz-Caraballo, Pérez, Rela '13.

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- Two weight setting Holmes, Lacey, Wick '16, also for biparameter Journé operators Holmes, Petermichl, Wick '17

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Theorem (Hytönen '12)

Let T be a Calderón-Zygmund operator, $w \in A_2$. Then there is a constant $C_{T,d} > 0$ such that for all $f \in L^2(w)$,

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As a corollary we conclude that for all Calderón-Zygmund operators T,

$$\|[T,b]f\|_{L^2(w)} \leq C_{T,d}\|b\|_{BMO}[w]_{A_2}^2 \|f\|_{L^2(w)}.$$

$$\|[T_b^k f\|_{L^2(w)} \leq C_{T,d}\|b\|_{BMO}^k[w]_{A_2}^{1+k} \|f\|_{L^2(w)}.$$

Chronology of first Linear Estimates on $L^2(w)$

- *Maximal function* (Buckley ‘93)
- *Martingale transform* (Wittwer ‘00)
- *Dyadic and continuous square function* (Hukovic,Treil,Volberg ‘00; Wittwer ‘02)
- *Beurling transform* (Petermichl, Volberg ‘02)
- *Hilbert transform* (Petermichl (’03) ‘07)
- *Riesz transforms* (Petermichl ‘08)
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Estimates based on Bellman functions and (bilinear) Carleson estimates (except for maximal function). The Bellman function method was introduced to harmonic analysis by Nazarov, Treil, Volberg (NTV). How about $L^p(w)$ estimates?

Sharp extrapolation d'après Rubio de Francia '82

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Theorem (Dragičević, Grafakos, P. , Petermichl '05)

If for all $w \in A_r$ there is $\alpha > 0$, and $C > 0$ such that

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then for each $1 < p < \infty$ and for all $w \in A_p$, there is $C_{p,r} > 0$

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Another proof Duoandikoetxea '11. Key are Buckley's '93 sharp bounds for the maximal function

$$\|Mf\|_{L^p(w)} \leq C_p[w]_{A_p}^{\frac{1}{p-1}}\|f\|_{L^p(w)}.$$

Beautiful proof by Lerner '08, better $A_p - A_\infty$ estimates HytPz '11, extensions to spaces of homogeneous type HytKairema '10.

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Example

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Sharp extrapolation from $r = 2$, $\alpha = 1$, is sharp for the martingale, Hilbert, Beurling-Ahlfors and Riesz transforms for all $1 < p < \infty$ (for $p > 2$ Petermichl, Volberg '02, '07, '08; $1 \leq p < 2$ DGPPet).

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Example

Extrapolation from linear bound in $L^2(w)$ is sharp for the dyadic square function only when $1 < p \leq 2$ ("sharp" DGPPet, "only" Lerner '07). However, extrapolation from square root bound on $L^3(w)$ is sharp (Cruz-Uribe, Martell, Pérez '12)

Some generalizations

- Off-diagonal and partial range extrapolation. Among the applications, they prove by iteration a multivariable extrapolation theorem and give a sharp bound for Calderón-Zygmund operators on $L^p(w)$ for weights in A_q $(q < p)$, Duoandicoetxea '11.

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- García-Cuerva, Rubio de Francia '85, and Cruz-Uribe, Martell, Pérez '11.

Dyadic vs Continuous Harmonic Analysis

- Martingale transform a dyadic toy model for CZ operators.

Dyadic vs Continuous Harmonic Analysis

- Martingale transform a dyadic toy model for CZ operators.
- Hilbert transform H, prototypical CZ operator, commutes with translations, dilations and anti-commutes with reflections. A linear and bounded operator T on $L^2(\mathbb{R})$ with those properties must be a constant multiple of the Hilbert transform: $T = cH$.

Dyadic vs Continuous Harmonic Analysis

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- Similarly for Beurling and Riesz transforms, and all CZ operators.
- Current Fashion: dominate (pointwise or else) all sorts of operators by sparse positive dyadic operators. Identifying the sparse collection involves using stopping-time techniques a favorite in the Garnett-Jones family!

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Properties

- *Nested:* $I,J\in\mathcal{D}$ then $I\cap J=\emptyset$, $\quad I\subseteq J$, or $J\subset I$.
- *One parent. if* $I\in\mathcal{D}_j$ *then there is a unique interval* $\tilde{I}\in\mathcal{D}_{j-1}$ *(the parent) such that* $I\subset\tilde{I}$, *and* $|\tilde{I}|=2|I|$.
- *Two children: There are exactly two disjoint intervals* $I_r,I_l\in\mathcal{D}_{j+1}$ *(the right and left children), with* $I=I_r\cup I_l$, $|I|=2|I_r|=2|I_l|$.

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Note: 0 *separates positive and negative dyadic interval, 2 quadrants.*

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The following parametrization will capture ALL dyadic grids on $\mathbb{R}$.

Lemma

For each scaling or dilation parameter r with $1 \leq r < 2$, and the random parameter β with $\beta = \{\beta_i\}_{i\in\mathbb{Z}}$, $\beta_i = 0, 1$, let $x_j = \sum_{i<-j} \beta_i 2^i$, the collection of intervals $\mathcal{D}^{r,\beta} = \cup_{j\in\mathbb{Z}} \mathcal{D}_j^{r,\beta}$ is a dyadic grid. Where

$$\mathcal{D}_j^{r,\beta} := r\mathcal{D}_j^{\beta}, \qquad \textit{and} \qquad \mathcal{D}_j^{\beta} := x_j + \mathcal{D}_j.$$

The advantage of this parametrization is that there is a very natural probability space, say $(\Omega, \mathbb{P})$ associated to the parameters, $\Omega = [1,2) \times \{0,1\}^{\mathbb{Z}}$. Averaging here means calculating the expectation in this probability space, that is $\mathbb{E}_{\Omega} f = \int_{\Omega} f(\omega)\, d\mathbb{P}(\omega)$.

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- Study of $T(b)$ theorems on metric spaces with non-doubling measures, NTV '97,'03, also Hyt, Martikainen '12.

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- BMO from dyadic BMO on the bidisc and product spaces of SHT Pipher, Ward '08, Chen, Li, Ward '13, inspired by celebrated Garnett and Jones '82.

Haar basis

Definition

Given an interval I, its associated *Haar function* is defined to be

$$h_I(x) := |I|^{-1/2}\big(\mathbb{1}_{I_r}(x) - \mathbb{1}_{I_l}(x)\big),$$

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Definition (The Martingale transform)

$$T_\sigma f(x) := \textstyle\sum_{I\in\mathcal{D}} \sigma_I \langle f, h_I\rangle h_I(x), \qquad \text{where} \quad \sigma_I = \pm 1.$$

Petermichl's dyadic shift operator

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Petermichl's dyadic shift operator Ш (pronounced "Sha") associated to the standard dyadic grid $\mathcal{D}$ is defined for functions $f \in L^2(\mathbb{R})$ by

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- More evidence comes from the way the family $\{Ш_{r,\beta}\}_{(r,\beta)\in\Omega}$ interacts with translations, dilations and reflections.

Petermichil's representation theorem for H

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- Result follows once one verifies that $c \neq 0$ (which she did!).
- $Ш_{r,\beta}$ are uniformly bounded on $L^p \Rightarrow$ Riesz's Theorem: H is bounded on L^p.
- Similar representation works for the *Beurling-Ahlfors* (Petermichl, Volberg '02), *Riesz transforms* (Petermichl '08).
- There is a representation valid for ALL Calderón-Zygmund singular integral operators (Hytönen '12).

Boundedness of H on weighted L^p

Theorem (Hunt, Muckenhoupt, Wheeden '73)

$$w \in A_p \Leftrightarrow \|Hf\|_{L^p(w)} \leq C_p(w)\|f\|_{L^p(w)}.$$

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$$\|Hf\|_{L^p(w)} \leq C_p[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}}\|f\|_{L^p(w)}.$$

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Sketch of the proof.

For $p = 2$ suffices to find uniform (on the grids) linear estimates for Petermichl's shift operator on $L^2(w)$. For $p \neq 2$ sharp extrapolation automatically gives the result from the *linear estimate* on $L^2(w)$. □

Two-weight problem for Hilbert transform

- Cotlar-Sadosky ‘80s à la Helson-Szegö.
- Various sets of sufficient conditions in between à la Muckenhoupt.
- Necessary and sufficient conditions Lacey, Sawyer, Shen, Uriarte-Tuero, and Lacey ‘14 . These are quantitative "Sawyer type" estimates.

Haar shift operators of arbitrary complexity

Definition (Lacey, Reguera, Petermichl '10)

A *Haar shift operator of complexity* (m, n) is

$$Ш_{m,n} f(x) := \sum_{L \in \mathcal{D}} \sum_{I \in \mathcal{D}_m(L), J \in \mathcal{D}_n(L)} c_{I,J}^L \langle f, h_I \rangle h_J(x),$$

where the coefficients $|c_{I,J}^L| \leq \frac{\sqrt{|I|\,|J|}}{|L|}$, and $\mathcal{D}_m(L)$ denotes the dyadic subintervals of L with length $2^{-m}|L|$.

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- The cancellation property of the Haar functions and the normalization of the coefficients ensures that $\|Ш_{m,n} f\|_2 \leq \|f\|_2$.
- T_σ is a Haar shift operator of complexity $(0, 0)$.
- $Ш$ is a Haar shift operator of complexity $(0, 1)$.
- The dyadic paraproduct π_b is not one of these.

The dyadic paraproduct

Definition

The *dyadic paraproduct* associated to $b \in BMO^d$ is

$$\pi_b f(x) := \sum_{I \in \mathcal{D}} m_I f \, \langle b, h_I \rangle h_I(x),$$

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- Paraproduct and adjoint are bounded operators in $L^p(\mathbb{R})$ if and only if $b \in BMO^d$. (A locally integrable function $b \in BMO^d$ iff for all $J \in \mathcal{D}$ there is $C > 0$ such that $\int_J |b(x) - m_J b|^2 dx = \sum_{I \in \mathcal{D}(J)} |\langle b, h_I \rangle|^2 \leq C|J|$.)

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- Formally, $fb = \pi_b f + \pi_b^* f + \pi_f b$.
- π_b bounded in $L^2(w)$ iff $w \in A_2$, moreover

$$\|\pi_b f\|_{L^2(w)} \le C[w]_{A_2} \|f\|_{L^2(w)} \qquad \text{(Beznosova '08)}.$$

Estimates for Shift Operators of arbitrary complexity

- Lacey, Petermichl, Reguera ('10) proved the A_2 conjecture for the Haar shift operators of arbitrary complexity (with constant depending exponentially in the complexity). Don't use Bellman functions. Use a *corona decomposition* and a *two-weight theorem for "well localized operators"* of NTV.

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- Hytönen '12 proved polynomial dependence.
- Paraproducts of arbitrary complexity Moraes, P. '13.

The A_2 conjecture (now Theorem)

Theorem (Hytönen 2010)

Let $1 < p < \infty$ *and let* T *be any Calderón-Zygmund singular integral operator in* $\mathbb{R}^n$, *then there is a constant* $c_{T,n,p} > 0$ *such that*

$$\|Tf\|_{L^p(w)} \leq c_{T,n,p}\, [w]_{A_p}^{\max\{1,\frac{1}{p-1}\}} \|f\|_{L^p(w)}.$$

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Sketch of the proof.

- Enough to show $p = 2$ thanks to sharp extrapolation.
- Prove a representation theorem in terms of Haar shift operators of arbitrary complexity and paraproducts on random dyadic grids.
- Prove linear estimates on $L^2(w)$ with respect to the A_2 characteristic for Haar shift operators and with polynomial dependence on the complexity (independent of the dyadic grid).

□

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Let T be a Calderón-Zygmund singular integral operator, then

$$Tf = \mathbb{E}_{\Omega}\left(\sum_{(m,n)\in\mathbb{N}^2} a_{m,n}\text{Ш}^{r,\beta}_{m,n}f + \pi^{r,\beta}_{T1}f + (\pi^{r,\beta}_{T^*1})^*f\right),$$

with $a_{m,n} = e^{-(m+n)\alpha/2}$, α is the smoothness parameter of T.

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- $Ш^{r,\beta}_{m,n}$ are Haar shift operators of complexity (m,n),
- $\pi^{r,\beta}_{T1}$ a dyadic paraproduct ($T1 \in BMO!$),
- $(\pi^{r,\beta}_{T^*1})^*$ the adjoint of a dyadic paraproduct ($T^*1 \in BMO!$).

All defined on random dyadic grid $\mathcal{D}^{r,\beta}$.

Case study: Dyadic proof for commutator $[H, b]$

Theorem (Daewon Chung '11)

$$\|[H, b]f\|_{L^2(w)} \leq C[w]_{A_2}^2 \|f\|_{L^2(w)}.$$

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$$\|[H,b]f\|_{L^2(w)} \leq C[w]_{A_2}^2 \|f\|_{L^2(w)}.$$

Daewon's "dyadic" proof is based on:

(1) the decomposition of the product bf

$$bf = \pi_b f + \pi_b^* f + \pi_f b$$

the first two terms are bounded in $L^p(w)$ when $b \in BMO$ and $w \in A_p$, the enemy is the third term.

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(3) Known linear bounds for paraproduct (Beznosova '08) and Ш (Petermichl '07).

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- The third term is better, it obeys a linear bound, and so do halves of the other two commutators (Chung '09, using Bellman):

$$\|Ш(\pi_f b) - \pi_{Ш f}(b)\| + \|Ш\pi_b f\| + \|\pi_b^* Ш f\| \leq C\|b\|_{BMO}[w]_{A_2}\|f\|$$

- Providing uniform (sharp) quadratic bounds for commutator $[Ш, b]$ hence averaging

$$\|[H, b]\|_{L^2(w)} \leq C\|b\|_{BMO}[w]_{A_2}^2\|f\|_{L^2(w)}.$$

Known to be sharp, bad guys are the non-local terms $\pi_b Ш$, $Ш\pi_b^*$.

cont. "dyadic proof" commutator

- A posteriori one realizes the pieces that obey linear bounds are generalized Haar Shift operators and hence their linear bounds can be deduced from general results for those operators ...
- As a byproduct of Chung's dyadic proof we get that Beznosova's extrapolated bounds for the paraproduct are optimal:

$$\|\pi_b f\|_{L^p(w)} \leq C_p [w]_{A_p}^{\max\{1, \frac{1}{p-1}\}} \|f\|_{L^p(w)}$$

Proof: by contradiction, if not for some p then $[H, b]$ will have better bound in $L^p(w)$ than the known optimal bound.

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Active area of research!

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- Domination by sparse positive dyadic operators: classical operators, Carleson operator, bilinear Hilber transform (multilinear multipliers), Hilbert transform along curves, oscillatory integrals...

Sparse positive dyadic operators

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bounded in $L^2(w)$ with linear bound when S is a sparse collection of dyadic intervals.

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- Example: If $b \in BMO$ then $\pi_b^* \pi_b$ is a bounded positive operator.

$$\pi_b^* \pi_b f(x) = \sum_{I \in \mathcal{D}} b_I^2 \, m_I f \, \frac{\mathbb{1}_I(x)}{|I|},$$

  The sequence $\{b_I^2\}_{I \in \mathcal{D}}$ is a Carleson sequence

$$\sum_{I \in \mathcal{D}(J)} b_I^2 \leq C|J|, \quad \forall J \in \mathcal{D}.$$

Sparse vs Carleson families of dyadic cubes

Definition

A collection of dyadic cubes $\mathcal{S}$ in $\mathbb{R}^d$ is η-sparse, $0 < \eta < 1$ if there are pairwise disjoint measurable sets

$$E_Q \subset Q \text{ with } |E_Q| \geq \eta|Q| \quad \forall Q \in \mathcal{S}.$$

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Lemma (Lerner-Nazarov in *Intuitive Dyadic Calculus*)

$\mathcal{S}$ is Λ-Carleson iff $\mathcal{S}$ is $1/\Lambda$-sparse.

Two weight problem for dyadic operators

- Necessary and sufficient conditions are known for the dyadic square function, martingale transform (NTV '99), well-localized dyadic operators (NTV '08) in the matrix context (Bickell, Culiuc, Treil, Wick arXiv '16). These are "testing or Sawyer" type conditions.

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Workshop on *Sparse domination of singular integral operators* October 9-13, 2017 at AIM organized by Amalia Culiuc, Francesco Di Plinio, and Yumeng Ou. Deadline for registration May 9th, today!

;-)

HAPPY BIRTHDAY PETER!!!! *Thanks Raanan, Chris, Ignacio, and specially Nam-Gyu for gathering us all in Seoul!!!*