Davis-Garsia Inequalities

Hardy Martingales in Banach Spaces

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Topics

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Review

Filtered probability space $(\Omega, (\mathcal{F}_n), \mathbb{P})$, Banach space X. An X valued martingale is a sequence of measurable functions $F_n : \Omega \to X$ satisfying

$$\mathbb{E}(F_n | \mathcal{F}_{n-1}) = F_{n-1}.$$

- Doob's maximal function estimate

$$\mathbb{E} \sup \|F_n\|^p \leq C_p \sup \mathbb{E} \|F_n\|^p, \quad 1 < p < \infty;$$

- The Davis decomposition, $F_n = G_n + B_n$ where

$$\|\Delta G_n\| \leq \max\{\|F_k\| : k \leq n-1\}, \quad \mathbb{E} \sum \|\Delta B_n\| \leq C \mathbb{E} \sup \|F_n\|$$

hold true for any Banach space X.

Conditions on the underlying Banach Space

Martingale inequalities imposing conditions on the underlying Banach space X.

- (Martingale cotype) There exists $2 \leq q < \infty$ such that

$$\mathbb{E}(\sum \|\Delta_k F\|_X^q)^{1/q} \leq C\mathbb{E} \sup \|F_k\|_X.$$

- (UMD) For every choice of signs $\pm$

$$\mathbb{E}\| \sum \pm \Delta_k F\|_X^p \leq C_p^p \sup_k \mathbb{E}\|F_k\|_X^p, \qquad 1 < p < \infty.$$

If $X = \mathbb{C}$, Martingale cotype holds true with $q = 2$, and UMD holds with $C_p \sim p^2/(p-1)$.

Banach spaces X for which Martingale Cotype resp. UMD hold true were intrinsically described by G. Pisier resp. D. Burkholder.

Hardy Martingales $H^1(\mathbb{T}^{\mathbb{N}}, X)$

$\mathbb{T}^{\mathbb{N}}$ the infinite torus-product with Haar measure $d\mathbb{P}$.

$F_k : \mathbb{T}^{\mathbb{N}} \to \mathbb{C}$ is $\mathcal{F}_k$ measurable iff

$$F_k(x) = F_k(x_1, \ldots, x_k), \quad x = (x_i)_{i=1}^{\infty}$$

Conditional expectation $\mathbb{E}_k F = \mathbb{E}(F|\mathcal{F}_k)$ is integration,

$$\mathbb{E}_k F(x) = \int_{\mathbb{T}^{\mathbb{N}}} F(x_1, \ldots, x_k, w) d\mathbb{P}(w).$$

An $(\mathcal{F}_k)$ martingale $F = (F_k)$ is a **Hardy martingale** if

$$y \to F_k(x_1, \ldots, x_{k-1}, y) \quad \in H^1(\mathbb{T}, X).$$

If $\Delta F_k(x_1, \ldots, x_{k-1}, y) = m_k(x_1, \ldots, x_{k-1}) y$ then (F_k) is called a **simple Hardy martingale**.

Complex analytic Hardy Spaces

$f \in L^p(\mathbb{T}, X)$, $\mathbb{T} = \{e^{i\theta} : |\theta| \leq \pi\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

The harmonic extension of f to the unit disk

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - |z|^2}{|z - e^{i\alpha}|^2} f(e^{i\alpha}) d\alpha, \qquad z \in \mathbb{D}.$$

Define $f \in H^p(\mathbb{T}, X)$ if $f \in L^p(\mathbb{T}, X)$ and the **harmonic extension of f is analytic** in $\mathbb{D}$.

Example: Maurey's embedding.

Fix $\epsilon > 0, w = (w_k) \in \mathbb{T}^{\mathbb{N}}$. Put $\varphi_1(w) = \epsilon w_1$, and

$$\varphi_n(w) = \varphi_{n-1}(w) + \epsilon(1 - |\varphi_{n-1}(w)|)^2 w_n.$$

Then $\lim |\varphi_n| = 1$ and $\varphi = \lim \varphi_n$ is uniformly distributed over $\mathbb{T}$.

For any $f \in H^1(\mathbb{T}, X)$

$$F_n(w) = f(\varphi_n(w)), \qquad w \in \mathbb{T}^{\mathbb{N}}$$

is an integrable Hardy martingale with **uniformly small** increments

$$\sup_{n\in\mathbb{N}} \mathbb{E}(\|F_n\|_X) = \int_{\mathbb{T}} \|f\|_X dm \quad \text{and} \quad \|\Delta F_n\|_X \leq 2\epsilon \int_{\mathbb{T}} \|f\|_X dm.$$

Pointwise estimates for ΔF_n.

Fix $w \in \mathbb{T}^{\mathbb{N}}, \quad n \in \mathbb{N}, \quad z = \varphi_n(w), \quad u = \varphi_{n-1}(w)$

$$\Delta F_n(w) = f(\varphi_n(w)) - f(\varphi_{n-1}(w)).$$

Cauchy integral formula

$$f(z) - f(u) = \int_{\mathbb{T}} \left\{ \frac{\zeta}{\zeta - z} - \frac{\zeta}{\zeta - u} \right\} f(\zeta) dm(\zeta).$$

Triangle inequality

$$\|f(z) - f(u)\|_X \leq \frac{|z-u|}{(1-|u|)(1-|z|)} \int_{\mathbb{T}} \|f\|_X dm$$

Example: Rudin Shapiro Martingales

Fix a complex sequence (c_n) with $\sum_{k=1}^{\infty} |c_k|^2 \leq 1$.

Define recursively: $F_1 = G_1 = 1$ and for $w = (w_n) \in \mathbb{T}^{\mathbb{N}}$

$$F_m(w) = F_{m-1}(w) + \overline{G_{m-1}}(w) c_m w_m,$$

$$G_m(w) = G_{m-1}(w) - \overline{F_{m-1}}(w) c_m w_m.$$

Pythagoras for (F_m, G_m) and $(\overline{G}_m, -\overline{F}_m)$ gives

$$|F_{m+1}(w)|^2 + |G_{m+1}(w)|^2 = (1 + |c_{m+1}|^2)(|F_m(w)|^2 + |G_m(w)|^2).$$

(F_n) uniformly bounded and $F_m(w) - F_{m-1}(w) = \overline{G_{m-1}}(w) c_m w_m$

(F_n) is a **simple Hardy martingale**

Maximal Functions estimate (Garling) For any X valued Hardy martingale (F_k),

$$\mathbb{E}(\sup_{k\in\mathbb{N}} \|F_k\|_X) \leq e \sup_{k\in\mathbb{N}} \mathbb{E}(\|F_k\|_X).$$

Davies Decomposition (PFXM) A Hardy martingale $F = (F_k)$ can be decomposed into Hardy martingales as $F = G + B$ such that

$$\|\Delta G_k\|_X \leq C\|F_{k-1}\|_X,$$

and

$$\mathbb{E}(\sum_{k=1}^{\infty} \|\Delta B_k\|_X) \leq C\mathbb{E}(\|F\|_X).$$

Lemma

If $h \in H_0^1(\mathbb{T}, X)$, $z \in X$ there exists $g \in H_0^\infty(\mathbb{T}, X)$ with

$$\|g(\zeta)\|_X \leq C_0 \|z\|_X, \qquad \zeta \in \mathbb{T}$$

and

$$\|z\|_X + \frac{1}{8} \int_{\mathbb{T}} \|h - g\|_X dm \leq \int_{\mathbb{T}} \|z + h\|_X dm.$$

Proof of Lemma (Sketch) . Let $\{z_t : t > 0\}$ denote complex Brownian Motion started at $0 \in \mathbb{D}$, and

$$\tau = \inf\{t > 0 : |z_t| > 1\}.$$

Define

$$\rho = \inf\{t < \tau : \|h(z_t)\|_X > C_0\|z\|_X\}.$$

- Doob's projection generates the analytic function

$$g(\zeta) = \mathbb{E}(h(z_\rho)|z_\tau = \zeta), \quad \zeta \in \mathbb{T}.$$

By choice of ρ, $\|h(z_\rho)\| \le C_0\|z\|_X$ and $\|g(\zeta)\|_X \le C_0\|z\|_X$.

Lower estimates for $\int_{\mathbb{T}} \|z + h\|_X dm$:

Put $A = \{\rho < \tau\}$. Define testing functions

$$p(\zeta) = \frac{1}{2}\mathbb{E}(1_A | z_\tau = \zeta), \qquad q = e^{\ln(1-p) + iH\ln(1-p)}.$$

- $\int_{\mathbb{T}} \|z + h\|_X p\,dm \geq (1/2 - 1/(2C_0))\mathbb{E}(\|h(z_\tau)1_A\|_X$
- $\int_{\mathbb{T}} \|z + h\|_X |q| dm \geq \|x\|_X (1 - 3\mathbb{P}(A))$
- $1 = p + |q|$.

$$\int_{\mathbb{T}} \|z + h\|_X dm \geq \|x\|_X + (1/2 - \delta(C_0))\mathbb{E}(\|h(z_\tau)1_A\|_X$$

- $\int_{\mathbb{T}} \|h - g\|_X dm \leq 2\mathbb{E}(\|h(z_\tau)1_A\|_X)$

Summing up:

$$\int_{\mathbb{T}} \|z + h\|_X dm \geq \|x\|_X + \delta \int_{\mathbb{T}} \|h - g\|_X dm$$

Sketch of Proof. Fix $x \in \mathbb{T}^{k-1}$. Put

$$h(y) = \Delta F_k(x,y) \quad \text{and} \quad z = F_{k-1}(x).$$

Lemma yields a bounded analytic g with

$$\|z\|_X + 1/8 \int_{\mathbb{T}} \|h - g\|_X dm \leq \int_{\mathbb{T}} \|z + h\|_X dm; \qquad \|g(\zeta)\|_X \leq C_0 \|z\|_X.$$

Define

$$\Delta G_k(x,y) = g(y), \qquad \Delta B_k(x,y) = h(y) - g(y).$$

Then

$$\|F_{k-1}\|_X + 1/8 \mathbb{E}_{k-1}(\|\Delta B_k\|_X) \leq \mathbb{E}_{k-1}(\|F_k\|_X).$$

Integrate and take the sum,

$$\sum \mathbb{E}(\|\Delta B_k\|_X) \leq 4 \sup \mathbb{E}(\|F_k\|_X).$$

The Davis decomposition yields vector valued Davis and Garsia inequalities for Hardy martingales. At this point a **hypothesis on the Banach space X is necessary :**

Let $q \geq 2$. A Banach space X satisfies the hypothesis $\mathcal{H}(q)$, if for each $M \geq 1$ there exists $\delta = \delta(M) > 0$ such that for any $x \in X$ with $\|x\| = 1$ and $g \in H_0^\infty(\mathbb{T}, X)$ with $\|g\|_\infty \leq M$,

$$\int_{\mathbb{T}} \|z + g\|_X dm \geq (1 + \delta \int_{\mathbb{T}} \|g\|_X^q dm)^{1/q}. \qquad (1)$$

• Condition (1) is required for uniformly bounded analytic functions g, and $\delta = \delta(M) > 0$ is allowed to depend on the uniform estimates $\|g\|_\infty \leq M$.

• If $X = \mathbb{C}$, the hypothesis "$\mathcal{H}(q)$" hold true with $q = 2$.

Theorem 1 *(PFXM) Let* $q \geq 2$*. Let* X *be a Banach satisfying* $\mathcal{H}(q)$*. There exists* $M > 0$ $\delta_q > 0$ *such that for any* $h \in H_0^1(\mathbb{T}, X)$ *and* $z \in X$ *there exists* $g \in H_0^\infty(\mathbb{T}, X)$ *satisfying*

$$\|g(\zeta)\|_X \leq M\|z\|_X, \qquad \zeta \in \mathbb{T},$$

and

$$\int_{\mathbb{T}} \|z+h\|_X dm \geq \left(\|z\|_X^q + \delta_q \int_{\mathbb{T}} \|g\|_X^q dm\right)^{1/q} + \frac{1}{16} \int_{\mathbb{T}} \|h-g\|_X dm.$$

The **Davis decomposition** and **hypothesis** $\mathcal{H}(q)$ **combined** give a decomposition of a H- mart. F into Hardy martingales such that $F = G + B$,

$$\|\Delta G_k\|_X \leq C\|F_{k-1}\|_X,$$

$$\mathbb{E}(\sum_{k=1}^{\infty}(\mathbb{E}_{k-1}\|\Delta G_k\|_X^q))^{1/q}+\mathbb{E}(\sum_{k=1}^{\infty}\|\Delta B_k\|_X) \leq A_q \sup_k \mathbb{E}\|F_k\|_X.$$

Use the above Theorem and **non- linear teleskoping:** Let $1 \leq q$, $1/p + 1/q$ and M_k, v_k non- negative such that

$$\mathbb{E}(M_{k-1}^q + v_k^q)^{1/q} \leq \mathbb{E}M_k \qquad \text{for } 1 \leq k \leq n, \tag{2}$$

then

$$\mathbb{E}(\sum_{k=1}^{n} v_k^q)^{1/q} \leq 2(\mathbb{E}M_n)^{1/q}(\mathbb{E}\max_{k\leq n} M_k)^{1/p} \tag{3}$$

Related Isomorphic Invariants: Assume that for every X valued Hardy martingale (F_k) we have:

• (ARNP) If $\sup_k \mathbb{E}\|F_k\| < \infty$ then (F_k) converges a.e.

• (Hardy martingale cotype) There exists $q < \infty$ such that

$$(\sum_k (\mathbb{E}\|\Delta_k F\|_X)^q)^{1/q} \leq C \sup_k \mathbb{E}\|F_k\|_X.$$

• (AUMD) For every choice of signs $\pm$

$$\mathbb{E}\|\sum \pm \Delta_k F\|_X \leq C \sup_k \mathbb{E}\|F_k\|_X.$$

AUMD and ARNP for a Banach space are already determined by testing **simple Hardy martingales.** This reduction is open for non trivial Hardy martingale Cotype.

Conditions implying Hardy martingale Cotype (HMC)

• There exists $\delta > 0$ such that for any $x \in X$ with $\|x\| = 1$ and $g \in H_0^1(\mathbb{T}, X)$,

$$\int \|z + g\|_X dm \geq (1 + \delta(\int \|g\|_X dm)^q)^{1/q}. \qquad (4)$$

•• For each $M \geq 1$ there exists $\delta = \delta(M) > 0$ such that for any $x \in X$ with $\|x\| = 1$ and $g \in H_0^\infty(\mathbb{T}, X)$ with $\|g\|_\infty \leq M$, the estimate (4) holds true

••• For any $x \in X$ with $\|x\| = 1$ and $g(\zeta) = \zeta w$, with $w \in X$ the estimate (4) holds true.

Condition • implies HMC. (directly)
By the methods discussed here •• implies HMC.
It is open whether ••• implies HMC.

The main sources

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