Bilinear operators and curvature

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Bilinear analog of the spherical averaging operator

- This talk is mainly about a bilinear analog of the spherical averaging operator and its variable coefficient generalizations. Define

$$B_\theta(f,g)(x) = \int_{S^{d-1}} f(x-y)g(x-\theta y)d\sigma(y),$$

where $\theta \in O_d(\mathbb{R})$ and σ is the surface measure on S^{d-1}.

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where $\theta \in O_d(\mathbb{R})$ and σ is the surface measure on S^{d-1}.

- We shall see that in a variety of ways it is a natural bilinear analog of the spherical averaging operator

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- We shall state and prove sharp $L^p(\mathbb{R}^2) \times L^q(\mathbb{R}^2) \to L^r(\mathbb{R}^2)$ for the spherical averaging operator in dimension two and describe the state of affairs in higher dimensions.

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- We shall state and prove sharp $L^p(\mathbb{R}^2) \times L^q(\mathbb{R}^2) \to L^r(\mathbb{R}^2)$ for the spherical averaging operator in dimension two and describe the state of affairs in higher dimensions.
- We shall state and prove the corresponding sharp bounds for the variable coefficient analog of the spherical averaging operator in dimension two.

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- is solved by

$$u(x,t) = ctA_t f(x).$$

A discrete analog of $A_tf(x)$

- Let P be a finite point set in $\mathbb{R}^d$, $d \geq 2$. Define a graph, called **the distance graph**, by taking points of P as vertices and connect two vertices x and y by an edge if $|x - y| = t$.

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$$Ef(x) = \sum_{|x-y|=t} f(y).$$

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- This viewpoint has been used by many authors in recent decades to study discrete problems using analytic methods.

Lebesgue space bounds for $A_t f(x)$

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- $$(0,0),\ (1,1)\ \left(\frac{d}{d+1}, \frac{1}{d+1}\right).$$
- The exponents are best possible as demonstrated by taking $f(x)$ to be the indicator function of a ball of radius δ, δ small.

Outline of the proof of Lebesgue space bounds for $A_t f(x)$

- Let
$$A_t^z f(x) = \frac{1}{\Gamma(z)} (1 - |\cdot|^2)_+^{z-1} \psi(\cdot) * f(x),$$
where ψ is a smooth cut-off function.

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- Stein's analytic interpolation theorem yields the claimed result.

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- and

$$det \begin{pmatrix} 0 & \nabla_x \phi \\ -(\nabla_y \phi)^T & \frac{\partial^2 \phi}{dx_i dy_j} \end{pmatrix} \neq 0$$

on the set $\{(x,y) : \phi(x,y) = t\}$.

Generalized Radon transform (continued)

- A result due to Phong and Stein says that under the assumptions above,

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$$\mathcal{R} : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$$

- for $\left(\frac{1}{p}, \frac{1}{q}\right)$ in the triangle with the endpoints $(0,0)$, $(1,1)$ and $\left(\frac{d}{d+1}, \frac{1}{d+1}\right)$, the same range as the spherical averaging operator.

L^p-improving measures

- In general, it is interesting to ask whether a given compactly supported Borel measure μ on $\mathbb{R}^d$ is L^p-improving in the sense that

$$||f * \mu||_{L^q(\mathbb{R}^d)} \leq C||f||_{L^p(\mathbb{R}^d)} \text{ for some } q > p.$$

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- For example, the Cantor-Lebesgue measure associated with the Cantor set of constant disection is L^p-improving (Christ; Oberlin).

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- $$= \sum_{y,z} K(y,z) f(x-y) g(x-z),$$

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- where K is the indicator function of the set

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- where θ is the rotation by $\frac{\pi}{3}$.
- This puts the discrete operator in the form

$$H(f,g)(x) = \sum_{|u|=t} f(x-u) g(x - \theta^{\pm} u).$$

The model bilinear generalized Radon transform

- This suggests that a reasonable model for the bilinear generalized Radon transform in $\mathbb{R}^2$ is

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- We will also discuss a natural variable coefficient analog of this operator in the style of the generalized Radon transforms studied by Phong, Stein and others in the 80s and and 90s.

Geometric configurations

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Theorem

Let $E \subset \mathbb{R}^d$, of positive upper Lebesgue density. Let E_δ denote the δ-neighborhood of E. Let $V = \{\mathbf{0}, v^1, v^2, \dots, v^{k-1}\} \subset \mathbb{R}^d$, where $k \geq 2$ is a positive integer. Then there exists $l_0 > 0$ such that for any $l > l_0$ and any $\delta > 0$ there exists $\{x^1, \dots, x^k\} \subset E_\delta$ congruent to $lV = \{\mathbf{0}, lv^1, \dots, lv^{k-1}\}$.

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- In $\mathbb{R}^2$ this result is, in general, false (Falconer). Chan, Laba and Pramanik proved that a positive result is possible if one assumes that $E \subset \mathbb{R}^2$ carries a measure with a decaying Fourier transform.

Bounds for the model operator

Theorem

(Greenleaf, Iosevich, Krause and Liu, 2016) With the notation above, $\theta \neq \pi$,

$$B_\theta : L^p(\mathbb{R}^2) \times L^q(\mathbb{R}^2) \to L^r(\mathbb{R}^2)$$

if $\left(\frac{1}{p}, \frac{1}{q}, \frac{1}{r}\right)$ *is in the polyhedron with the vertices* $(0,0,0)$, $(\frac{2}{3}, \frac{2}{3}, 1)$, $(0, \frac{2}{3}, \frac{1}{3})$, $(\frac{2}{3}, 0, \frac{1}{3})$, $(1,0,1)$, $(0,1,1)$ *and* $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

If $\theta = \pi$, *the vertices of the polyhedron are* $(0,0,0)$, $(0, \frac{2}{3}, \frac{1}{3})$. $(\frac{2}{3}, 0, \frac{1}{3})$, $(1,0,1)$, $(0,1,1)$ *and* $(\frac{2}{3}, \frac{2}{3}, 1)$.

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- These exponents are best possible, at least on the scale of normed spaces, $p, q, r \geq 1$.

Typical difficulties with bilinear operators

- Let $T_K f(x) = \int f(x-y)K(y)dy$. Then Plancherel tells us that

$$T_K : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \text{ if and only if } \widehat{K} \in L^\infty(\mathbb{R}^d).$$

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- Let $B_K(f,g)(x) = \int \int f(x-u)g(x-v)K(u,v)dudv$. The corresponding natural estimate in this setting is

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- It is known that $\widehat{K} \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ does not, in general, guarantee that this estimate holds. Just take $T_K(f,g)(x) = H(fg)(x)$, where H is the Hilbert transform.

The bilinear multiplier of the model operator

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- This multiplier does not decay at all along the 2-plane

$$\xi+\theta^T\eta = 0.$$

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- $$=: \int f(x) \cdot (T_\theta g)(x)dx$$

The $L^{\frac{3}{2}}(\mathbb{R}^2) \times L^{\frac{3}{2}}(\mathbb{R}^2) \to L^1(\mathbb{R}^2)$ bound (continued)

- $$\leq ||f||_{L^{\frac{3}{2}}(\mathbb{R}^2)} \cdot ||T_\theta g(x)||_{L^3(\mathbb{R}^2)}$$

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- The same argument yields the $L^{\frac{d+1}{d}}(\mathbb{R}^d) \times L^{\frac{d+1}{d}}(\mathbb{R}^d) \to L^{d+1}(\mathbb{R}^d)$ bound for $d \geq 2$.

The $L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ bound

- Let

$$S_\theta = \left\{ (y, y') \in S^1 \times S^1 : |\alpha - \alpha'| < \frac{\theta}{2} \right\}, \text{ with } y = e^{i\alpha}, y' = e^{i\alpha'}.$$

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- $$u_1 = x_1 - \cos(\alpha),\ \ u_2 = x_2 - \sin(\alpha),$$ $$v_1 = x_1 - \cos(\alpha' + \theta), v_2 = x_2 - \sin(\alpha' + \theta).$$

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$$\sin(\alpha - \alpha' - \theta),$$

The $L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ bound (concluded)

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- It follows that

$$||B_\theta(\chi_E, \chi_F)||_{L^2(\mathbb{R}^2)} \leq C(|E|^2 + |E||F|)^{\frac{1}{2}} \leq 2C|E|^{\frac{1}{2}}|F|^{\frac{1}{2}}.$$

for some constant C depending only on θ.

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- $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{q},$ (Large ball).

The general case

- Let

$$B(f,g)(x) = \int\int \delta(\phi_1(x,y)-t_1)\cdot\delta(\phi_2(x,z)-t_2)\cdot\delta(\phi_3(y,z)-t_3)\,dydz.$$

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- We need a **curvature** condition and a **transversality** condition.

Geometric conditions

- The **curvature** assumption is just the rotational curvature condition on ϕ_3, namely

$$det \begin{pmatrix} 0 & \nabla_x \phi_3 \\ -(\nabla_y \phi_3)^T & \frac{\partial^2 \phi_3}{dx_i dy_j} \end{pmatrix} \neq 0$$

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- $$det \begin{bmatrix} d_y \phi_1(x,y) & 0 \\ d_y \phi_3(y,z) & 0 \\ 0 & d_{z'} \phi_3(x,z') \\ 0 & d_{z'} \phi_3(y,' z') \end{bmatrix} \neq 0,$$

Geometric conditions (continued)

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