Many Theorems and a Few Stories

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OUTLINE

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I. Extension Theorems for BMO and Sobolev Spaces.

$\Omega \subset \mathbb{R}^d$ connected and open, $\varphi : \Omega \to \mathbb{R}$,

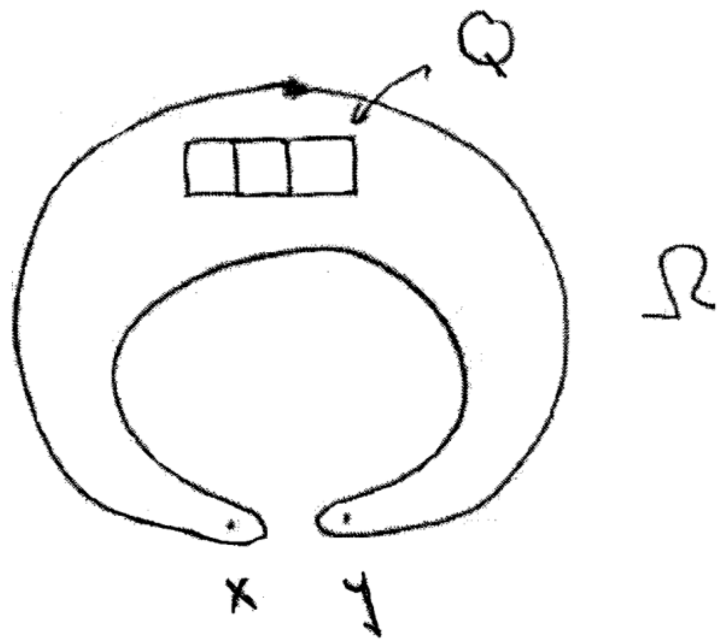
$$||\varphi||_{BMO(\Omega)} = \sup_{Q \subset \Omega} \frac{1}{|Q|} \int_Q |\varphi - \varphi_Q| dx$$

where Q is a $|Q| =$ its measure, $\varphi_Q = \frac{1}{|Q|} \int_Q \varphi dx$.

Theorem 1: Every $\varphi \in BMO(\Omega)$ has extension in $BMO(\mathbb{R}^d)$ if and only if for all $x, y \in \Omega$

$$\inf_{\Omega \supset \gamma \text{ joins } x,y} \int_\gamma \frac{ds(z)}{\delta(z)} \leq C \left| \log \frac{\delta(x)}{\delta(y)} \right| + C \log\left(2 + \frac{|x-y|}{\delta(x) + \delta(y)}\right),$$

where $\delta(x) = \text{dist}(x, \partial\Omega)$.


Q
Ω
x
y

Corollary: $\Omega \subset \mathbb{R}^2$ and $\partial\Omega = \Gamma$ a Jordan curve.

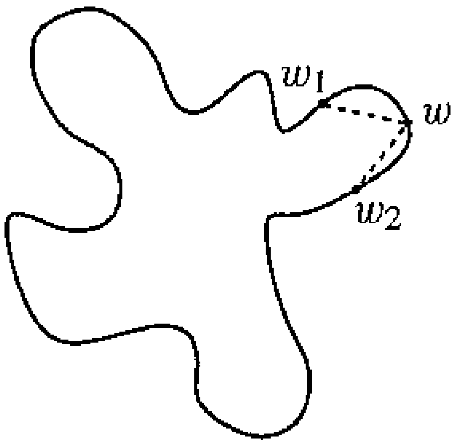
$$\text{Every } \varphi \in BMO(\Omega) \text{ has extension in } BMO(\mathbb{R}^2)$$

$$\Updownarrow$$

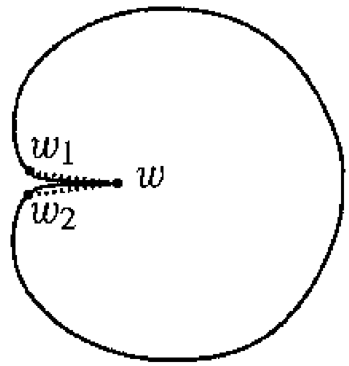
$$|w_1 - w_3| \leq C|w_1 - w_2|$$

for $w_1, w_2 \in \Gamma$ and w_3 on the smaller arc (w_1, w_2),

i.e. if and only if Γ is a quasicircle.



Quasicircle



Not Quasicircle

$$L^p_k(\Omega) = \{f \in L^p(\Omega) : |\alpha| \leq k => D^\alpha f \in L^p(\Omega)\},$$

for $1 \leq p \leq \infty$, $k \in \mathbb{N}$.

Theorem 2: (Acta 1981) For any k, and p there exists a bounded linear extension operator

$$\Lambda_k : L^p_k(\Omega) \to L^p_k(\mathbb{R}^n)$$

if and only if $\exists \varepsilon > 0$, $0 < \delta \leq \infty$ so that Ω is an (ε, δ) **domain**:

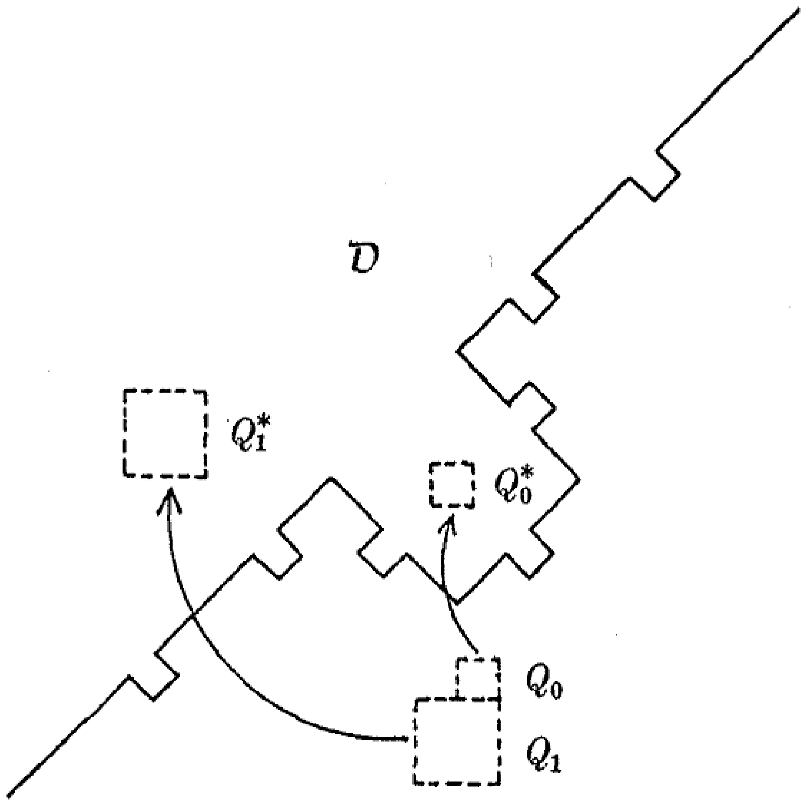
$$x, y \in \Omega, |x - y| < \delta$$

$$\Downarrow$$

$$\exists \text{ arc } \gamma \subset \Omega \text{ joining } x, y \text{ with length}(\gamma) \leq \frac{\varepsilon}{|x - y|}$$

and

$$\text{dist}(z, \partial\Omega) \geq \varepsilon \frac{|x - z||y - z|}{|x - y|}, \quad \forall z \in \gamma.$$



$\mathcal{D}$
Q_1^*
Q_0^*
Q_0
Q_1

II. BMO and A_p Weights.

John-Nirenberg Theorem: $\varphi \in BMO(\mathbb{R}^d) \Leftrightarrow \ \exists\, c:$

$$\sup_Q \frac{1}{|Q|} \int_Q e^{c|\varphi(x) - \varphi_Q|} dx < \infty. \tag{JN}$$

Theorem 3: (Annals 1978) If $\varphi \in BMO(\mathbb{R}^d)$, then

$$\inf_{g \in L^\infty} ||\varphi - g||_{BMO} \sim \sup\{c : \text{ JN holds}\}.$$

A weight $w \geq 0$ on $\mathbb{R}^n$ is an A_p-weight $1 \leq p < \infty$ if

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q w dx\Big)\Big(\frac{1}{|Q|}\int_Q w^{\frac{-1}{p-1}} dx\Big) < \infty.$$

(holds if and only if singular integrals or H-L maximal operator is bounded on $L^p(w)$.)

Theorem 4: (Annals 1980)

$$w \in A_p \Leftrightarrow w = w_1 w_2^{1-p}, w_1, w_2 \in A_1.$$

Theorem 4 $\Longrightarrow$ Theorem 3.

See also J. L. Rubio de Francia, Annals of Math 1982, for an elegant non-constructive proof of Theorem 4.

$Q \subset \mathbb{R}^d$ is a **dyadic cube** if $\exists\, n, k_j \in \mathbb{Z}$ so that

$$Q = \bigcap_{j=1}^{d} \{k_j 2^{-n} \le x_j \le (k_j + 1)2^{-n}\}.$$

$\varphi \in L^1_{\text{loc}}$ is BMO_{d} if

$$||\varphi||_{BMO_{\text{d}}} = \sup_{Q \text{ dyadic}} \frac{1}{|Q|} \int_Q |\varphi - \varphi_Q| dx < \infty.$$

$BMO \subset BMO_{\text{d}}$, $BMO \neq BMO_{\text{d}}$, but BMO_{d} was a simpler space.

Theorem 5: (Pacific J. 1982). Assume

$$\mathbb{R}^d \ni \alpha \to \varphi^{(\alpha)} \in BMO_{\text{d}}$$

is measurable, $||\varphi^{(\alpha)}||_{BMO_{\text{d}}} \le 1$, $\varphi^{(\alpha)}_{Q_o} = 0$ for a fixed Q_o and all α. Then

$$\varphi(x) = \lim_{N \to \infty} \frac{1}{(2N)^d} \int_{|\alpha_j| \le N} \varphi^{(\alpha)}(x + \alpha) d\alpha$$

is BMO and $||\varphi||_{BMO} \le C_d$.

Theorem 5 yields BMO theorems like Theorem 3 from their simpler dyadic counterparts. For related H^1 result, see B. Davis, TAMS 1980.

Let $w \in L^1(\mathbb{R}), w \geq 0$. Then

$$\sup_I \Big(\frac{1}{|I|}\int_I w dx\Big)\Big(\frac{1}{|I|}\int_I \frac{1}{w} dx\Big) < \infty \tag{A_2}$$

holds if and only if w satisfies the Helson-Szegö condition:

$$w = e^{u+\tilde{v}}, \quad u \in L^\infty, ||v||_\infty < \frac{\pi}{2}, \tag{HS}$$

because both hold $\Leftrightarrow$ Hibert transform is $L^2(w)$ bounded.

In dimension 1, A_2 and HS imply Theorem 3.

Problem: Prove $A_2 \implies$ HS directly, without using the $L^2(w)$ boundedness of H or M.

III. Constructions with H^∞ Interpolation, $\overline{\partial}$, and BMO.

Let $\{z_j\}$ be a sequence in the upper half plane $\mathbb{H} = \{x + iy : y > 0\}$ and

$$H^\infty = \{f : \mathbb{H} \to \mathbb{C} : f \text{ is bounded and analytic}\}.$$

Theorem (Carleson 1958) Every interpolation problem

$$f(z_j) = a_j,\ \ j = 1, 2, \dots,\ \ (a_j) \in \ell^\infty \tag{INT}$$

has solution $f \in H^\infty$ if and only if

(i) $\inf_{k \neq j} \frac{|z_j - z_k|}{y_j} \geq c > 0$ (hyperbolic separation)

and

(ii) for all intervals $I \subset \mathbb{R}$,

$$\sum_{x_j \in I, y_j < |I|} y_j \leq C|I|,$$

(Carleson measure condition).

Problems: (already solved) Find constructive solutions to:

(1) INT

(2) $\varphi \in BMO(\mathbb{R}) \Longrightarrow \varphi = u + Hv, u, v \in L^\infty$

(3) μ Carleson measure on $\mathbb{H}$:

$$\mu(I \times (0, |I|]) \leq ||\mu||_C |I|$$

$$\Downarrow$$

$$\overline{\partial F} = \mu$$

has solution on $\mathbb{H}$ which is bounded on $\mathbb{R}$.

Theorem 6: (Annals 1980) Constructive solutions to (2) and (3).

Proof uses:

(i) the J. P. Earl solution to (1),

(ii) Approximation of Carleson measures by measures $\sum_{y_j} \delta_{z_j}$ from interpolating sequences $\{z_j\}$, and

(iii) a BMO extension theorem of Varopoulos.

For another construction, define for σ a measure on $\mathbb{H}$:

$$K(\sigma, z, \zeta) = \frac{2i}{\pi} \frac{\text{Im}\zeta}{(z-\zeta)(z-\overline{\zeta})} \exp\Bigg(\int_{\text{Im}w \leq \text{Im}\zeta} \Big(\frac{i}{\zeta - \overline{w}} - \frac{i}{z - \overline{w}} \Big) d|\sigma|(w) \Bigg).$$

Theorem 7: (Acta Math., 1983) If μ is a Carleson measure on $\mathbb{H}$, then

$$S(\mu)(z) = \int_{\mathbb{H}} K\big(\frac{\mu}{||\mu||_C}, z, \zeta\big) d\mu(\zeta) \in L^1_{\text{loc}}$$

satisfies

$$\overline{\partial} S(\mu) = \mu \quad \text{on } \mathbb{H},$$

and

$$\sup_{\mathbb{R}} |S(\mu)(x)| \leq C_0 ||\mu||_C.$$

Theorem 8: Let $\{z_j\} \subset \mathbb{H}$ satisfy

(i) $\inf_{k \neq j} \frac{|z_j - z_k|}{y_j} \geq c > 0$ (hyperbolic separation)

and

(ii) for all intervals $I \subset \mathbb{R}$, $\sum_{x_j \in I, y_j < |I|} y_j \leq C|I|$.

Define

$$B_j(z) = \prod_{k; k \neq j} \alpha_k \frac{z - z_k}{z - \overline{z_k}},$$

where $|\alpha_k| = 1$ are convergence factors, and

$$\delta = \inf_j |B_j(z_j)| > 0.$$

Then

$$F_j(z) = \gamma_j B_j(z) \Big(\frac{y_j}{z - \overline{z_j}} \Big)^2 \exp\left(\frac{-i}{\log 2/\delta} \sum_{y_k \leq y_j} \frac{y_k}{z - \overline{z_k}} \right),$$

in which

$$\gamma_j = \frac{-4}{B_j(z_j)} \exp\left(\frac{i}{\log 2/\delta} \sum_{y_k \leq y_j} \frac{y_k}{z_j - \overline{z}_k} \right),$$

satisfies

$$4F_j(z_k) = \delta_{j,k} \;\text{ and }\; \sum |F_j(z)| \leq C_0 \frac{\log 2/\delta}{\delta}.$$

Paul Koosis called this "the Peter Jones mechanical interpolation formula".

IV. Corona Theorems and Problems.

When $H^\infty(\Omega)$ is the algebra of bounded analytic functions on a complex manifold Ω, the **corona problem** for Ω is: Given $f_1, \dots, f_n \in H^\infty(\Omega)$ such that for all $z \in \Omega$,

$$\max_{1 \le j \le n} |f_j(z)| \ge \delta > 0$$

are there $g_1, \dots, g_n \in H^\infty(\Omega)$ such that

$$f_1 g_1 + \dots f_n g_n = 1?$$

$\Omega =$ unit disc $\mathbb{D}$, Yes, Carleson (1962).

Ω a finite bordered Riemann surface, Yes, E. L. Stout (1964), many later proofs.

Ω a Riemann surface, No, Brian Cole (ca 1970).

Problem: Ω an infinitely connected plane domain.

Theorem: (Carleson (1983)) If $\mathbb{C} \setminus \Omega = E \subset \mathbb{R}$ and for all $x \in E$

$$|E \cap [x - r, x + r]| \geq cr,$$

then the corona theorem holds for Ω.

Forelli Projection: $\Omega = \mathbb{D}/\Gamma$,

(i) $P : H^\infty(\mathbb{D}) \to H^\infty(\Omega) = \{f \in H^\infty(\mathbb{D}) : f \circ \gamma = f, \forall \gamma \in \Gamma\}$;

(ii) $||P(f)||_\infty \leq C||f||_\infty$;

(iii) $P(fg) = fP(g), \quad f \in H^\infty(\Omega)$;

(iv) $P(1) = 1$.

Forelli Projection $\Rightarrow$ corona theorem for Ω. Carleson built a Forelli Projection.

Theorem 9: (Jones and Marshall) Let $G(z, \zeta)$ be Green's function for Ω, fix z_0 and let $\{\zeta_k\}$ be the critical points of $G(z_0, \zeta)$. If there is $A > 0$ such that all components of

$$\{\zeta \in \Omega : \sum_k G(\zeta, \zeta_k) > A\}$$

are simply connected, then Ω has a projection operator and the corona theorem holds for Ω.

For $\mathbb{C} \setminus \Omega \subset \mathbb{R}$, Theorem 9 $\Longrightarrow$ Carleson's theorem.

Theorem 10: If $\mathbb{C} \setminus \Omega \subset \mathbb{R}$, the corona theorem holds for Ω.

Note: $|E| = 0 \Longleftrightarrow H^\infty(\Omega)$ trivial.

Proof of Theorem 10 uses constructions from both Theorem 6 and Theorem 8.

Problem: Corona theorem for $\Omega = \mathbb{C} \setminus E$, $E \subset \Gamma$, a Lipschitz graph.

Known if Γ is $C^{1+\varepsilon}$, or if $\Lambda_1(E \cap B(z, r)) \geq cr \ \forall z \in E$.

Problem: Corona theorem for $\mathbb{C} \setminus (K \times K)$, $K = \frac{1}{3}$ Cantor set.

Problem: Which $\Omega \subset \mathbb{C}$ have Forelli Projections?

For $\mathbb{C} \setminus \Omega = E \subset \mathbb{R}$, it holds $\Longleftrightarrow |E \cap [x - r, x + r]| \geq cr \quad \forall x \in E$.

V. Harmonic Measure and Integral Mean Spectra.

Theorem: (Makarov, 1985) Let Ω be a simply connected plane domain and ω harmonic measure for $z_0 \in \Omega$. Then

$$\alpha < 1 \Rightarrow \omega << \Lambda_\alpha$$

$$\alpha > 1 \Rightarrow \omega \perp \Lambda_\alpha.$$

For a bounded univalent function φ define

$$\beta_\varphi(t) = \inf\Big\{\beta : \int_0^{2\pi} |\varphi'(re^{i\theta})|^t = O((1-r)^{-\beta})\Big\}$$

and the **integral mean spectrum**,

$$B(t) = \sup_\varphi \big\{\beta_\varphi(t)\big\}.$$

Makarov's Theorem is $\Leftrightarrow B(0) = 0$.

Brennan's Conjecture is $B(-2) = 1$

With $\varphi = \sum_{n=1}^{\infty} a_n z^n$, write $A_n = \sup_{||\varphi||_\infty \leq 1} |a_n|$.

Theorem 11:(Carleson-Jones, Duke J. 1992) For bounded φ the limit

$$\gamma = -\lim_{n\to\infty} \frac{\log A_n}{\log n}$$

exists and there exists bounded φ_1 such that

$$\gamma = -\lim_{n\to\infty} \frac{\log a_n}{\log n}.$$

Moreover, $1 - \gamma = B(1)$.

Carleson and Jones further conjectured $\gamma = \frac{3}{4}$, i.e. $B(1) = \frac{1}{4}$. Belyaev proved $\gamma < .78$, i.e. $B(1) \geq .23$.

Brennan-Carlson-Jones-Kraetzer Conjecture: $B(t) = \frac{t^2}{4}, |t| \leq 2$.

Theorem 12: (Jones-Makarov, Annals 1995)

$$B(t) = t - 1 + O((t-2)^2) \quad (t \to 2).$$

For arbitrary plane domains Jones and Wolff proved:

Theorem 13: (Acta 1988) Let Ω be a plane domain such that $\partial\Omega$ has positive logarithmic capacity. Then there exists $F \subset \partial\Omega$ of Hausdorff dimension ≤ 1 and $\omega(z, F) = 1$ for $z \in \Omega$.

Proof uses classical potential theory and the formula

$$\frac{1}{2\pi}\int_{\partial\Omega} \frac{\partial G}{\partial n} \log \frac{\partial G}{\partial n} dx = \gamma = \sum_{\nabla G(\zeta_j)=0} G(\zeta_j)$$

from Ahlfors used earlier by Carleson (to show dim ω stricly less than dim $\partial\Omega$ in certain cases).

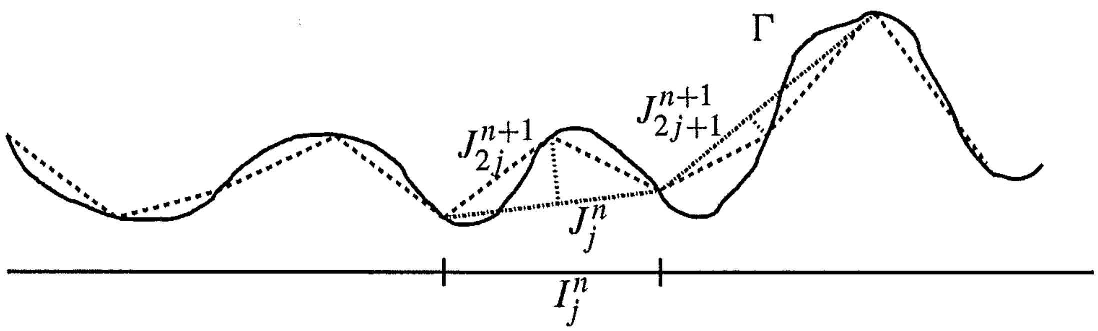
VI. Traveling Salesman Theorem.

Geometric Lemma: (SLLM 1384) Let $\Gamma = \{\gamma(x) : x \in [0,1]\}$ be a Lipschitz graph in $\mathbb{R}^2$. For a dyadic interval $I \subset \mathbb{R}$ set

$$\beta_\Gamma(I) = \frac{1}{|I|} \inf_L \sup_{x \in I} \operatorname{dist}(\gamma(x), L),$$

Then

$$\sum_{I \subset J} \beta_\Gamma^2(I)|I| \leq C\Lambda_1(\Gamma).$$


Γ
J_{2j+1}^{n+1}
J_{2j}^{n+1}
J_j^n
I_j^n

For bounded $K \subset \mathbb{R}^2$ and Q a dyadic square of side $\ell(Q)$ in $\mathbb{R}^2$, let $w(Q)$ be the width the narrowest strip containing $K \cap 3Q$ and

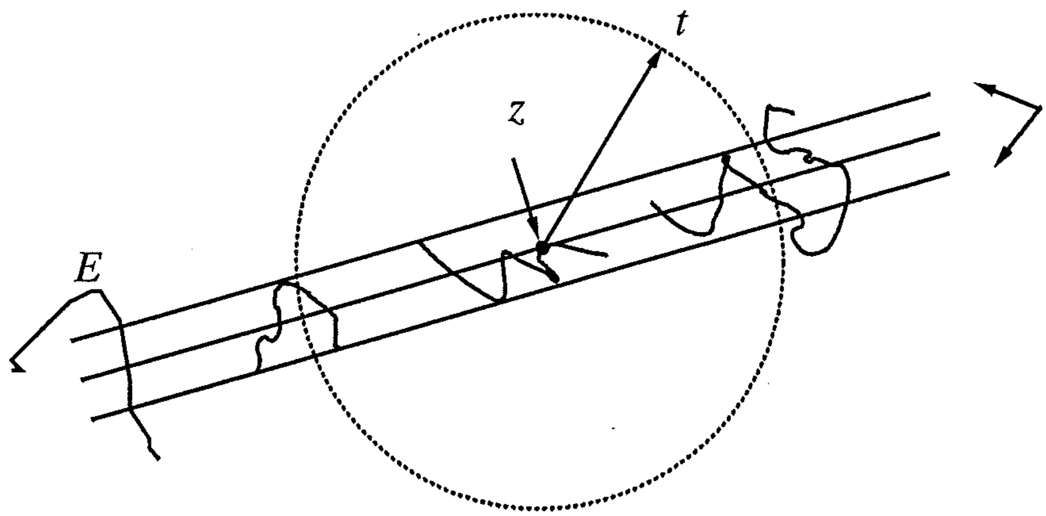
$$\beta_K(Q) = \frac{w(Q)}{\ell(Q)}.$$

Theorem 14: There exists a rectifiable curve $\Gamma \supset K$ if and only if

$$\beta^2(K) = \sum_Q \beta_K^2(Q)\ell(Q) < \infty.$$

Moreover,

$$\Lambda_1(\Gamma) \leq C\big(\text{diam}K + \beta^2(K)\big).$$


t
z
E

VII. Work with Bishop: Harmonic Measure and Kleinian Groups.

Let Γ be a rectifable curve in $\mathbb{C}$, Ω a simply connected domain, and $\varphi : \mathbb{D} \to \Omega$ a conformal mapping.

Theorem 15: (Annals 1990) On $\Gamma \cap \partial\Omega$, Ω-harmonic measure is absolutely continuous to linear measure:

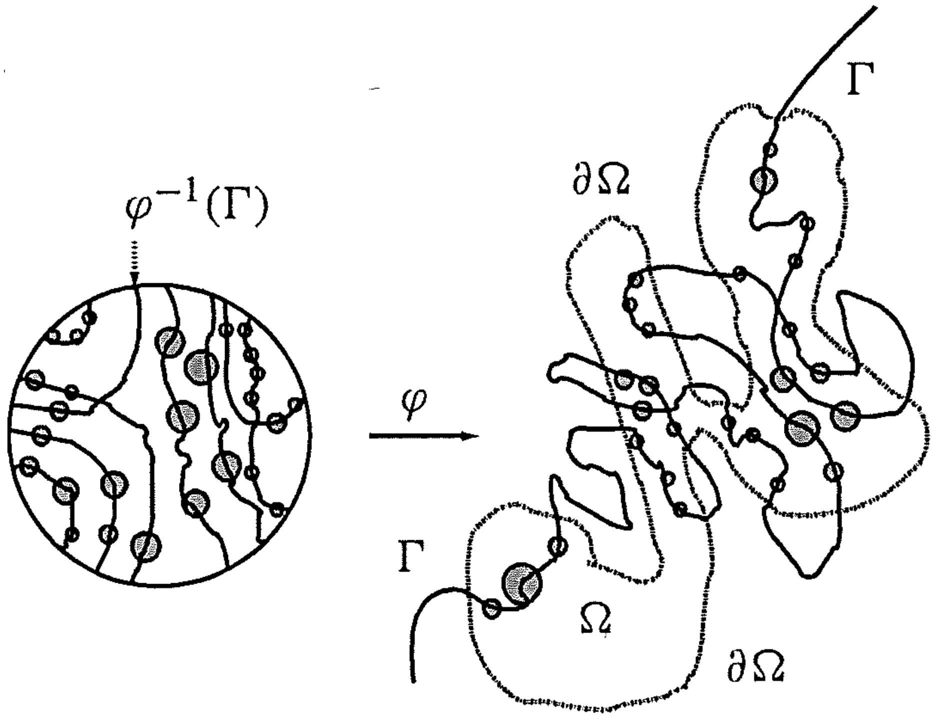
$$E \subset \Gamma \cap \partial\Omega, \text{ and } \omega(z, E, \Omega) > 0 \Rightarrow \Lambda_1(E) > 0.$$

Proof uses Theorem 14 and a related estimate on the Schwarzian derivative.

Γ is **Ahlfors regular** if $\Lambda_1(\Gamma \cap B(z,r)) \leq Mr$ for all $z \in \Gamma$.

Theorem 16: Γ is Ahlfors regular if and only if there is C_Γ such that for all Ω and $\varphi : \mathbb{D} \to \Omega$,

$$\Lambda_1(\varphi^{-1}(\Gamma \cap \Omega)) \leq C_\Gamma.$$


$\varphi^{-1}(\Gamma)$
φ
Γ
$\partial\Omega$
Γ
Ω
$\partial\Omega$

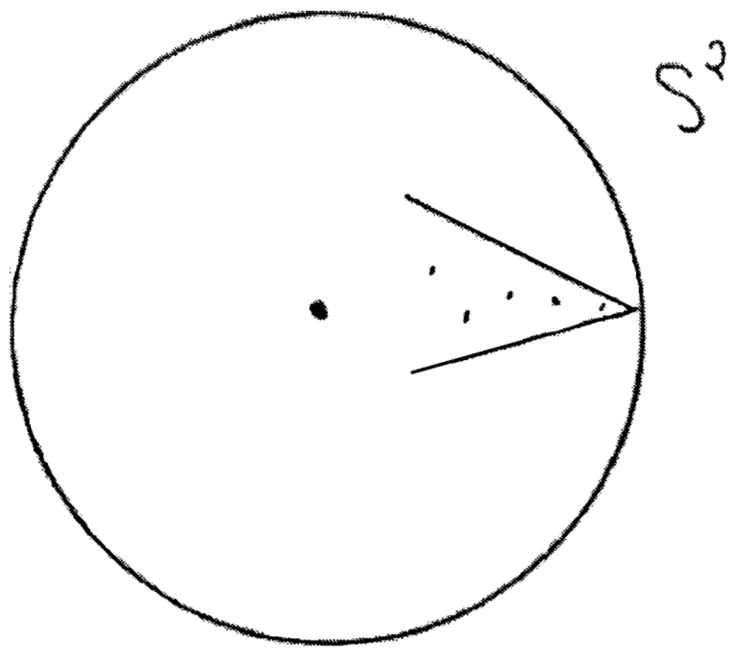
A **Kleinian group** is a discrete group G of Möbius transformations acting on S^2 (and the hyperbolic 3-ball) B such that the **limit set** $\Lambda(G)$ (accumulation points of the orbit $\{\gamma(0) : \gamma \in G\}) \neq S^2$.

The **Poincaré exponent**

$$\delta(G) = \inf\Big\{ s : \sum_{G} e^{-\rho_B(0,\gamma(0))} < \infty \Big\}$$

measures the speed at which $\gamma(0)$ tends to $S^2 = \partial B$.

The **conical limit set** of G is $\Lambda_c(G) \subset \Lambda(G)$ consists of the nontangential accumulation points of the orbit.


S²

Theorem 17:(Acta 1997) If $\Lambda(G)$ is infinite, then

$$\dim_{\text{Hausd}}\bigl(\Lambda_0(G)\bigr) = \delta(G).$$

G is **geometrically finite** if some finite-sided hyperbolic polygon in B is a fundamental domain.

G is **analytically finite** if $\Omega(G)/G$ is a finite union of compact surfaces minus finitely many points.

Theorem 18: If G is analytically finite but not geometrically finite, then

$$\dim_{\text{Hausd}}\bigl(\Lambda(G)\bigr) = 2.$$

VIII. Applied Mathematics.

Jones, Maggioni and Schul construct local coordinates on a domain in $\mathbb{R}^d$ (or on a C^α manifold) using Laplace eigenfunctions:

Dirichlet or Neumann eigenfunctions $\{\varphi_j\}$ for Δ on Ω with $|\Omega| < \infty$,

$$0 \leq \lambda_0 \leq \cdots \leq \lambda_j \leq \dots$$

$$\#\{j : \lambda_j \leq T\} \leq C_W T^{d/2} |\Omega|.$$

Theorem 19: (PNAS 2008) Assume $|\Omega| = 1$. There are constants $c_1, \dots, c_6$ (depending on d and C_W) so that for $z \in \Omega$ and $R = R_z = \mathrm{dist}(z, \partial\Omega)$, there exist indices $j_1, \dots, j_d$ and constants $c_6 R \leq \gamma_1 \dots \gamma_d \leq 1$ so that

$$B_R(z) \ni x \to \Phi(x) = \big(\gamma_1 \varphi_{j_1}(x), \dots \gamma_d \varphi_{j_d}(x)\big)$$

satisfies

$$\frac{c_1}{R} ||x_1 - x_2|| \leq ||\Phi(x_1) - \Phi(x_2)|| \leq \frac{c_2}{R} ||x_1 - x_2||$$

for $x_1, x_2 \in B_{c_1 R}(z)$, and the corresponding eigenvalues satisfy

$$\frac{c_4}{R^2} \leq \lambda_{j_k} \leq \frac{c_5}{R^2}.$$

IX. Random Welding.

Let $\Gamma \subset \mathbb{C}$ be a Jordan curve bounding domains $\Omega_\pm$ and let $f_\pm : \mathbb{D}_\pm \to \Omega_\pm$ be conformal. Then $\varphi = f_+^{-1} \circ f_- : \mathbb{T} \to \mathbb{T}$ is the **welding map**.

Welding Problem: Characterize welding maps.

Beurling-Ahlfors: (1956) φ quasisymmetric $\Rightarrow \exists$ welding, but Γ is a quasicircle.

Theorem 20 (Astala, Jones, Kupiainen, Saksman) Let

$$\varphi(e^{2\pi it}) = e^{2\pi ih(t)},$$

where

$$h(t) = \frac{\tau([0,t))}{\tau([0,1)]},$$

and τ is the random measure

$$d\tau = e^{\beta X(t)}dt$$

with $0 \le \beta < \sqrt{2}$ and

$$X(t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}(A_n \cos 2\pi nt + B_n \sin 2\pi nt)$$

where A_n, B_n are i.i.d. $N(0,1)$ Gaussians. Then almost surely φ is a Hölder continuous circle homeomorphism and φ is the welding for a Jordan curve $\Gamma = \partial f_+(\mathbb{D})$, f_+ and Γ are Hölder continuous, and Γ is unique up to Möbius tranformations.

Notes: Almost surely, Γ is not a quasicircle.

In proof X is replaced by a white noise approximation X_ε.

Uniqueness follows from Hölder continuity and a theorem of Jones and Smirnov.

Existence uses Lehto's solution of the Beltrami equation $f_{\overline{z}} = \mu f_z$ for degenerate μ and three giant steps:

(1) The (1956) Beurling-Ahlfors extension of φ to $f : \mathbb{D} \to \mathbb{D}$ and a careful analysis of images $f(Q)$, $Q \subset \mathbb{D}$ a Whitney cube.

(2) Sharp probalistic estimates for $\frac{\tau(J)}{\tau(J')}$ for adjacent dyadic intervals $J, J' \subset [0, 1)$.

(3) A representation of Gaussian free field $X(t)$ due to Barcy and Muzy.

Thank you.