

# CLE EXTREME NESTING & LIOUVILLE QUANTUM GRAVITY

**Bertrand Duplantier**

Institut de Physique Théorique

Université Paris-Saclay, France

GEOMETRY, ANALYSIS AND PROBABILITY

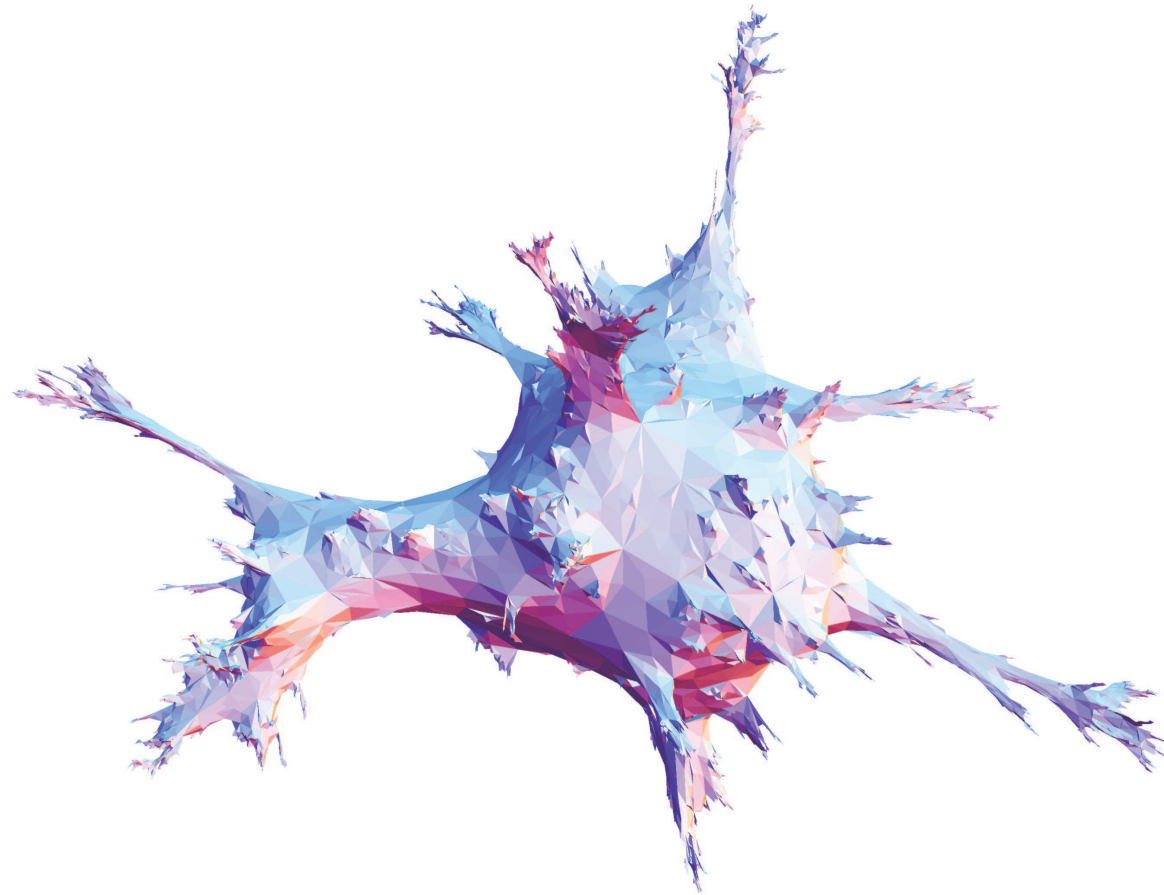
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*Joint work with*

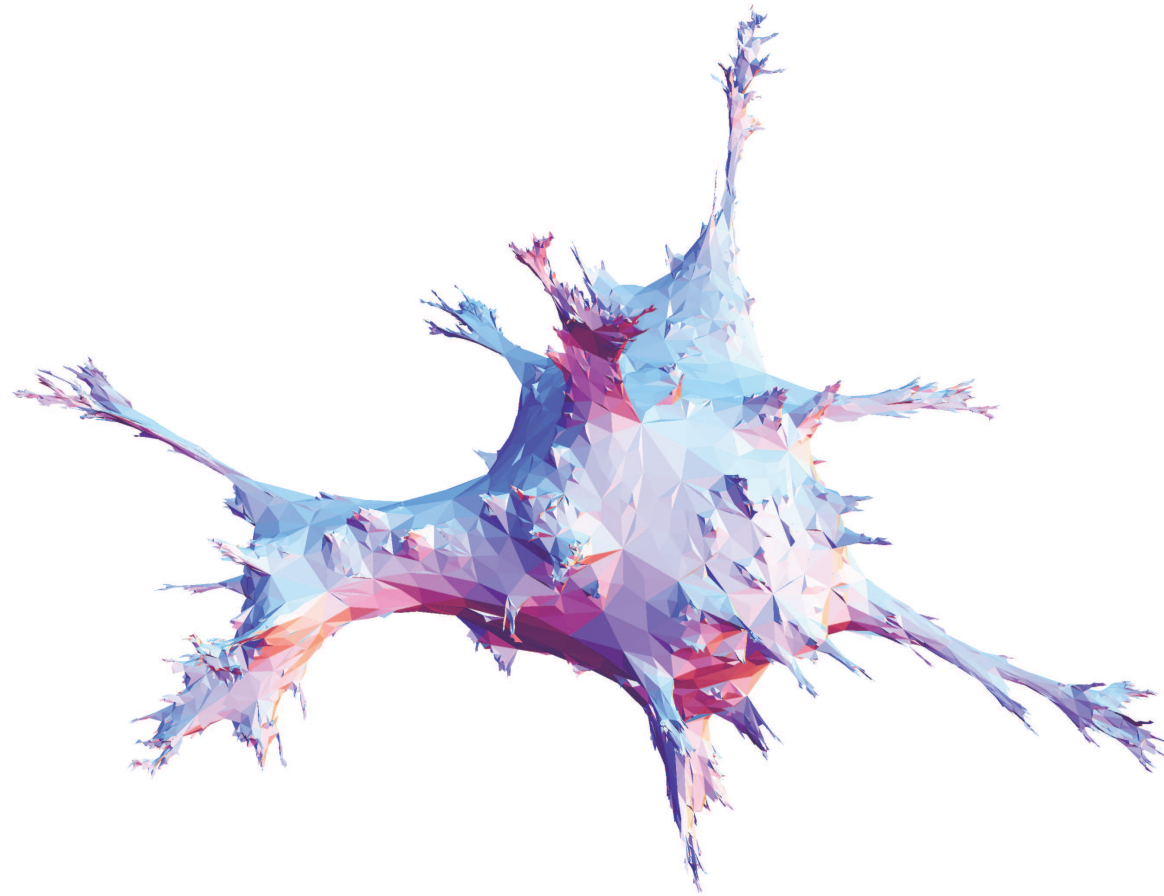
- Gaëtan Borot (MPI Bonn) & Jérémie Bouttier (ENS-Lyon)

# Random Planar Map



*A random triangulation [Courtesy of N. Curien].*

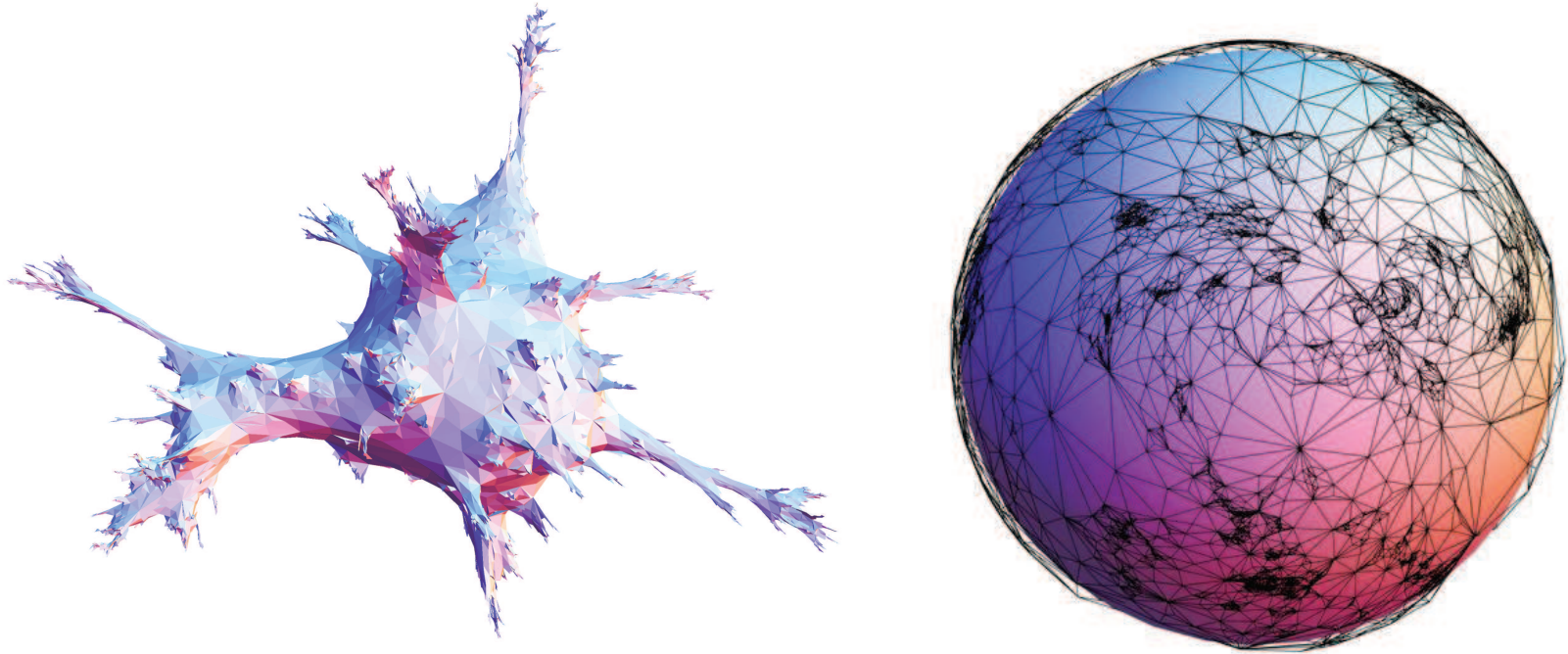
# Random Planar Map



*A random triangulation [Courtesy of N. Curien].*

*Continuum limit: **The Brownian Map** [Le Gall '11; Miermont '11]*

# Random Planar Map & Conformal Map



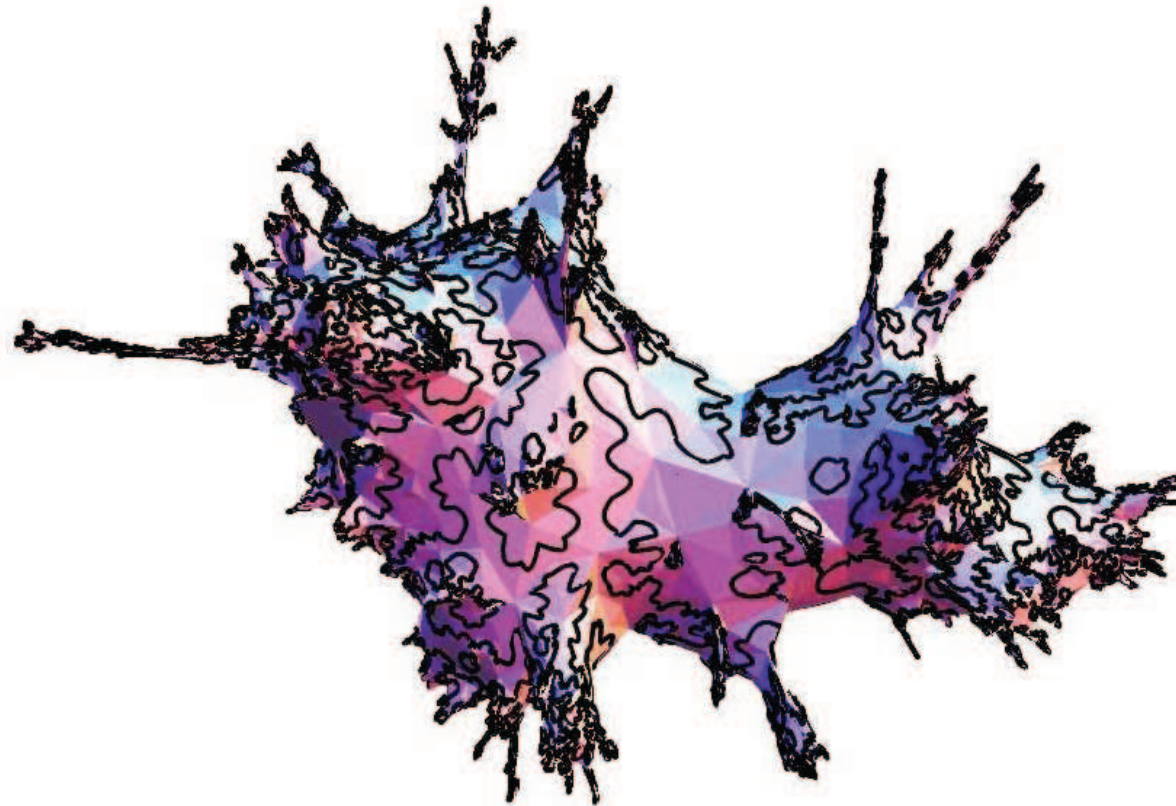
*[Courtesy of N. Curien]*

*Left: A random triangulation of the sphere. Right: Conformal map to the sphere.*

*In the continuum scaling limit: **Liouville Quantum Gravity***

*A.M. Polyakov '81*

# Random Planar Map & Statistical Model



*Percolation hulls [Courtesy of N. Curien].*

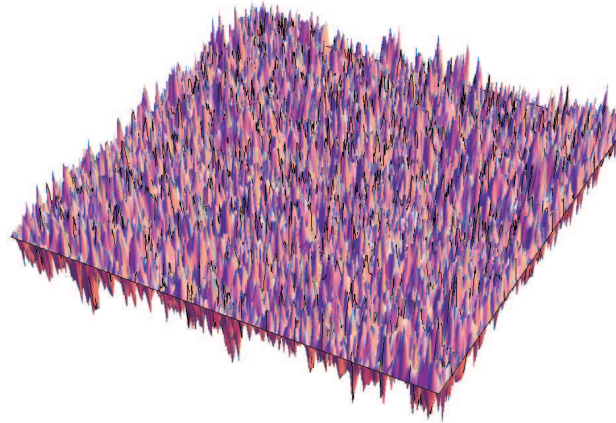
# LIOUVILLE QG

## RANDOM MEASURE

$$\mu = "e^{\gamma h} dz"$$



# Gaussian Free Field (GFF)



[Courtesy of J. Miller]

*Distribution*  $h$  with *Gaussian weight*  $\exp\left[-\frac{1}{2}(h, h)_{\nabla}\right]$ , and **Dirichlet inner product** in domain  $D$

$$\begin{aligned}(f_1, f_2)_{\nabla} &:= (2\pi)^{-1} \int_D \nabla f_1(z) \cdot \nabla f_2(z) dz \\ &= \text{Cov}\left((h, f_1)_{\nabla}, (h, f_2)_{\nabla}\right)\end{aligned}$$

# LIIOUVILLE QUANTUM MEASURE

$$\mu_\gamma := \lim_{\varepsilon \rightarrow 0} \exp[\gamma h_\varepsilon(z)] \varepsilon^{\gamma^2/2} dz,$$

where  $h_\varepsilon(z)$  is the GFF average on a circle of radius  $\varepsilon$ ;  
converges weakly for  $\gamma < 2$  to a random measure, denoted by  
 $\mu_\gamma = e^{\gamma h(z)} dz$ , and singular w. r. t. Lebesgue measure.

[Høegh-Krohn '71; Kahane '85; D. & Sheffield '11]

For  $\gamma = 2$ , the renormalized one,

$$\sqrt{\log(1/\varepsilon)} \left[ \exp[\gamma h_\varepsilon(z)] \varepsilon^{\gamma^2/2} \right]_{\gamma=2} dz,$$

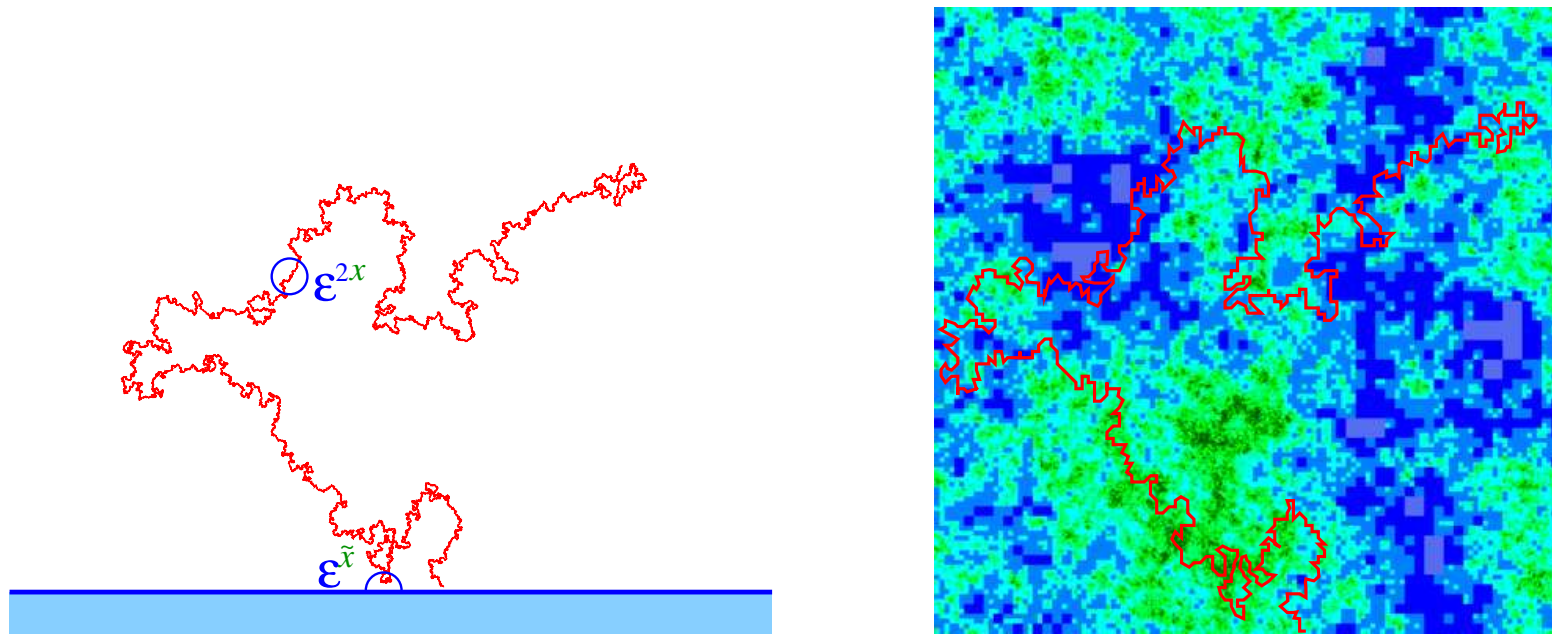
converges, as  $\varepsilon \rightarrow 0$ , to a positive *non-atomic* random  
measure.

[D., Rhodes, Sheffield, Vargas '14]



# Scaling Exponents of (Random) Fractals

SAW in half plane - 1,000,000 steps



[Courtesy of T. Kennedy & J. Miller]

*Probabilities & Hausdorff Dimensions (e.g., SLE<sub>κ</sub>)*

$$\mathbb{P} \asymp \varepsilon^{2x}, \quad \tilde{\mathbb{P}} \asymp \varepsilon^{\tilde{x}}, \quad d = 2 - 2x \quad (= 1 + \kappa/8)$$

$$\delta\text{-Quantum Ball:} \quad \mathbf{P} \asymp \delta^\Delta, \quad \tilde{\mathbf{P}} \asymp \delta^{\tilde{\Delta}}$$

KNIZHNIK, POLYAKOV, ZAMOLODCHIKOV '88

$x$  and  $\Delta$  ( $\tilde{x}$  and  $\tilde{\Delta}$ ) are related by the **KPZ formula**

$$x = U_{\gamma}(\Delta) := \left(1 - \frac{\gamma^2}{4}\right) \Delta + \frac{\gamma^2}{4} \Delta^2$$

*Kazakov '86; D. & Kostov '88 [Random matrices]*

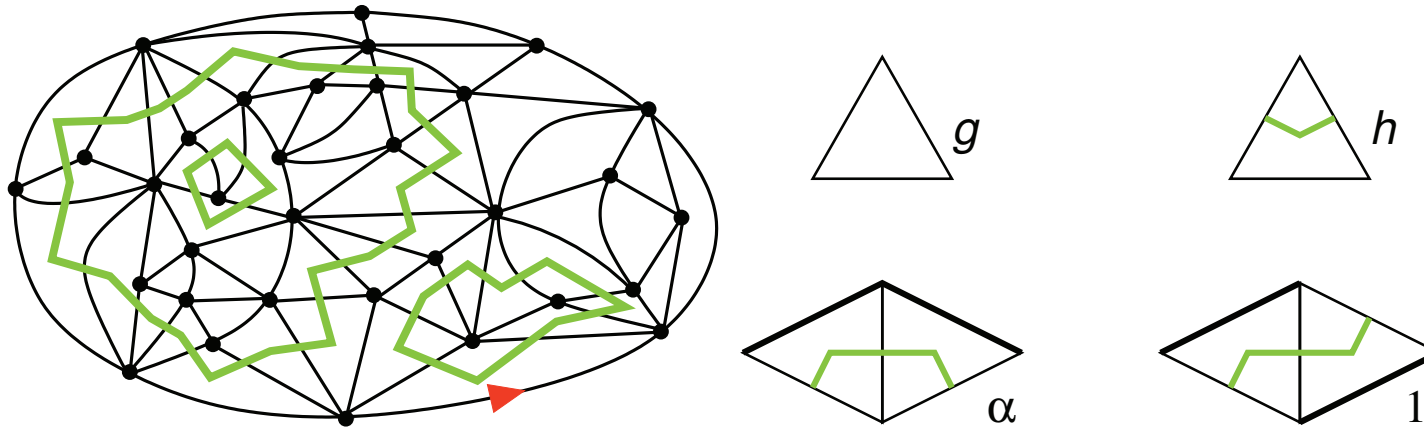
*David; Distler & Kawai '88 [Liouville field theory]*

**KPZ Theorem** – *D. & Sheffield '11*

*Benjamini & Schramm '09; Rhodes & Vargas '11 [Hausdorff dimension]*

*David & Bauer '09; Berestycki, Garban, Rhodes, Vargas '14 [Heat kernel]*

# $O(n)$ -Loop Model on a Random Planar Map

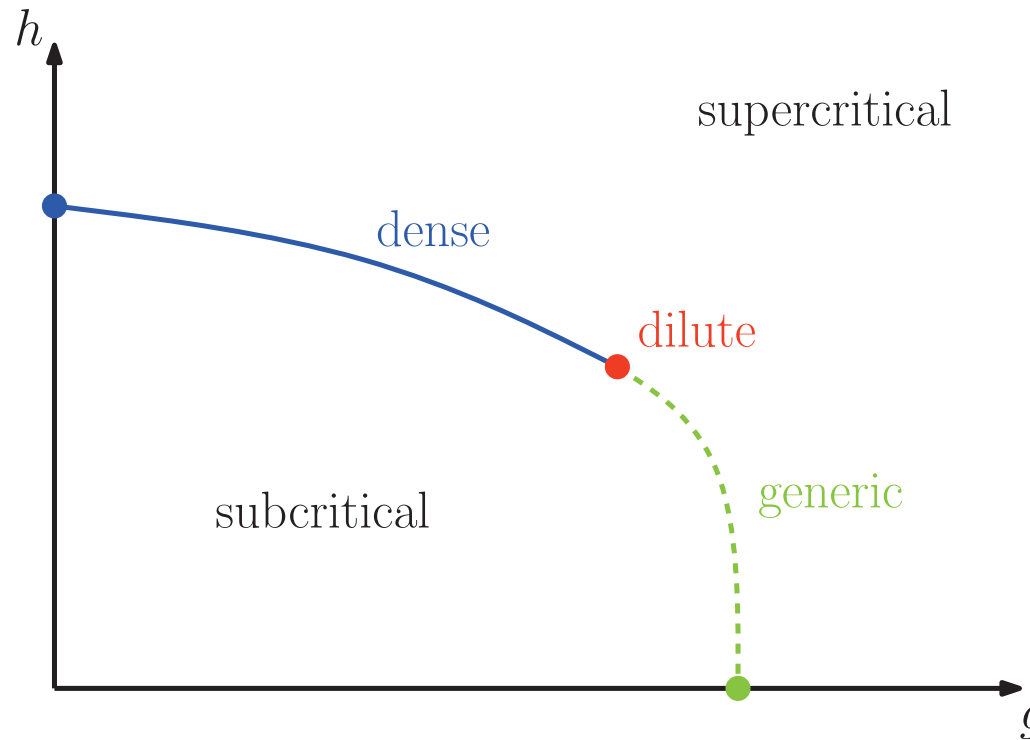


Disk triangulation and local weights ( $\alpha = 1$ ).

$$Z_\ell = \sum_{\mathcal{C}} u^{V(\mathcal{C})} w(\mathcal{C}), \quad w(\mathcal{C}) = n^{\mathcal{L}} g^{T_1} h^{T_2};$$

- Sum over all **configurations**  $\mathcal{C}$  of a **disk** of fixed **perimeter**  $\ell$
- $u$  auxiliary weight per **vertex**,  $V(\mathcal{C})$  total number of vertices (**volume**)
- $T_1, T_2$  numbers of empty or **occupied** triangles
- number of **loops**  $\mathcal{L}$  of  $\mathcal{C}$  weighted by  $n \in [0, 2]$ .

# Phase Diagram



Phase diagram of the  $O(n)$ -loop model ( $n \in [0, 2]$ ) on a random map. For  $u = 1$ , a line of critical points separates the subcritical and supercritical phases. Critical points may be in three different **universality classes**: **generic**, **dilute** and **dense**.

## Random Map $O(n)$ Nesting Theorem [Borot, Bouttier, D. '16]

Fix  $(g, h)$  and  $n \in (0, 2)$  such that the model reaches a **dilute** or **dense** critical point for the vertex weight  $u = 1$ . In the ensemble of random pointed disks of volume  $V$  and perimeter  $L$ , the probability distribution of the **number  $\mathcal{N}$  of separating loops** between the marked point and the boundary behaves for **large  $V$**  as:

$$\mathbb{P}\left[\mathcal{N} = \frac{c \ln V}{\pi} p \mid V, L = \ell\right] \sim (\ln V)^{-\frac{1}{2}} V^{-\frac{c}{\pi} J(p)} \quad (\text{sphere}),$$

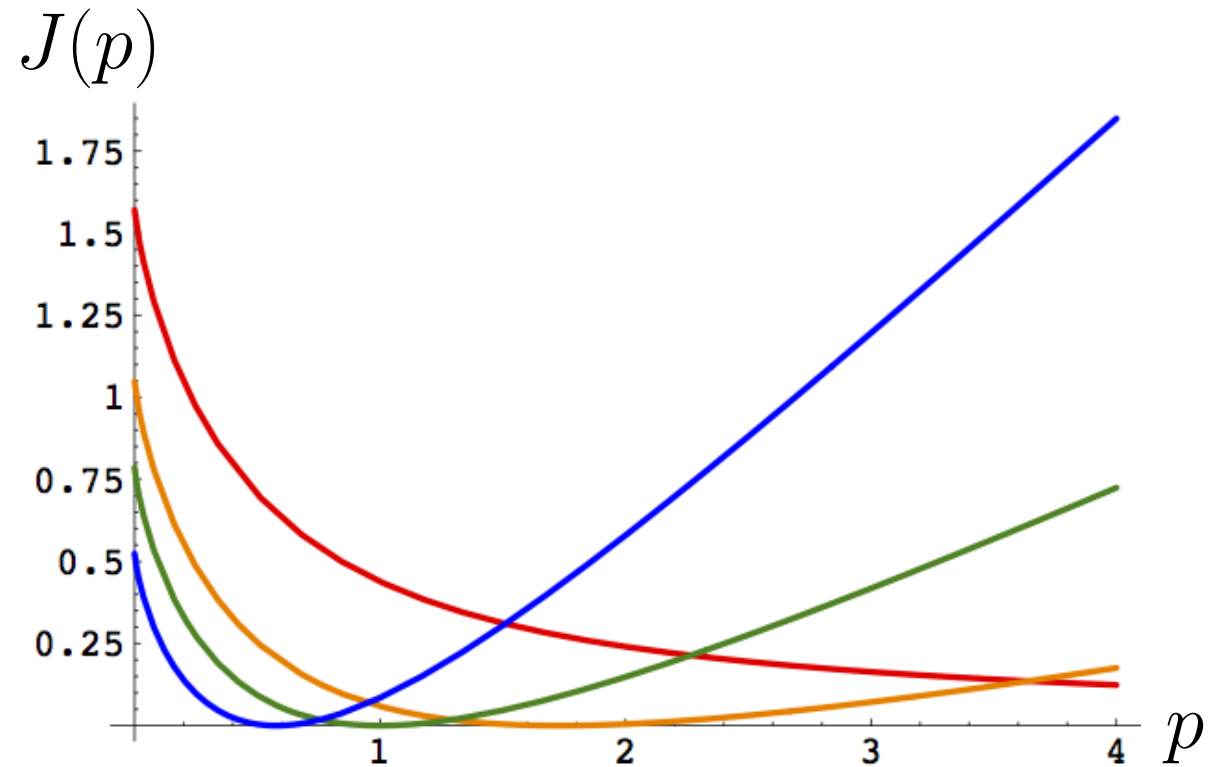
$$\mathbb{P}\left[\mathcal{N} = \frac{c \ln V}{2\pi} p \mid V, L = V^{\frac{c}{2}} \ell\right] \sim (\ln V)^{-\frac{1}{2}} V^{-\frac{c}{2\pi} J(p)} \quad (\text{disk}),$$

with  $\ell > 0$  fixed, and the **large deviations function**

$$J(p) = p \ln \left( \frac{2}{n} \frac{p}{\sqrt{1+p^2}} \right) + \operatorname{arccot}(p) - \arccos(n/2);$$

$c = 1$  (**dilute**) or  $c = 1/[1 - \frac{1}{\pi} \arccos(\frac{n}{2})]$  (**dense**) decreases from 2 to 1 as  $n$  increases from 0 to 2.

# Large Deviations Function



$J(p)$  for  $n = 1$  (Ising & Percolation),  $n = \sqrt{2}$  (FK Ising),  
 $n = \sqrt{3}$  (3-state Potts),  $n = 2$  (4-state Potts &  $CLE_4$ ).

# The Conformal Loop Ensemble (CLE)

[Sheffield '09, Sheffield & Werner '12]

The **critical  $O(n)$ -model** on a regular planar lattice is predicted to converge in the continuum limit to  $SLE_{\kappa}/CLE_{\kappa}$ , for

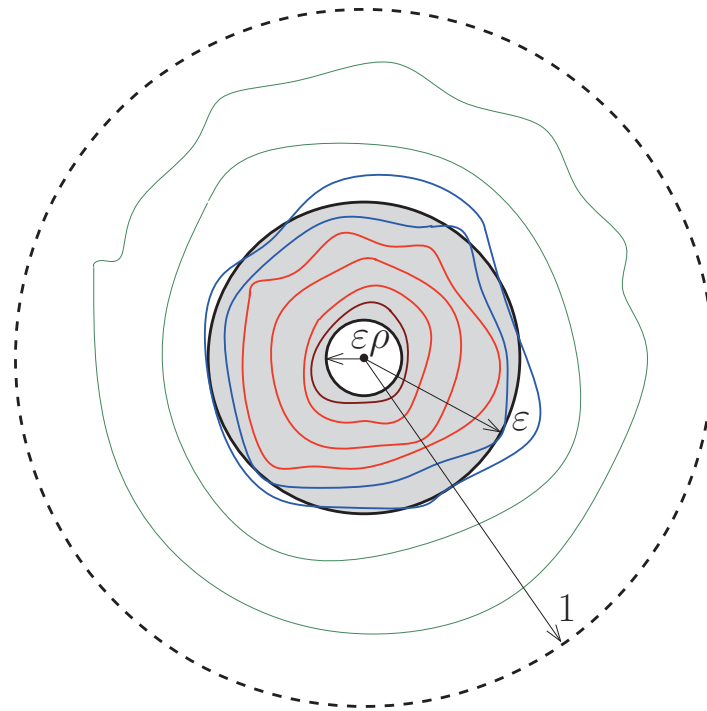
$$n = -2 \cos(4\pi/\kappa), \quad n \in (0, 2], \quad \begin{cases} \kappa \in (8/3, 4], & \text{dilute phase} \\ \kappa \in [4, 8), & \text{dense phase,} \end{cases}$$

(Loop-erased random walk & spanning trees [Lawler, Schramm, Werner], Ising & percolation [Smirnov], GFF contour lines [Schramm-Sheffield].)

The same is expected for the  **$O(n)$ -model** on a **random planar map**, the random area measure becoming in the critical scaling limit the **Liouville quantum measure**  $\mu_{\gamma}$  for

$$\gamma = \min\{\sqrt{\kappa}, 4/\sqrt{\kappa}\}.$$

# Nesting in the Conformal Loop Ensemble (CLE)



$\mathcal{N}_z(\epsilon)$  is the number of nested loops of a  $CLE_\kappa$ ,  $\kappa \in (8/3, 8)$  surrounding the ball  $B(z, \epsilon)$  in the unit disk.



## Extreme nesting in CLE *[Miller, Watson & Wilson '14]*

Let  $\mathcal{N}_z(\varepsilon)$  be the number of loops of a  $\text{CLE}_\kappa$ ,  $\kappa \in (8/3, 8)$  surrounding the ball  $B(z, \varepsilon)$ , and  $\Phi_{\mathbf{v}}$  the set of points  $z$  where

$$\lim_{\varepsilon \rightarrow 0} \mathcal{N}_z(\varepsilon) / \ln(1/\varepsilon) = \mathbf{v}.$$

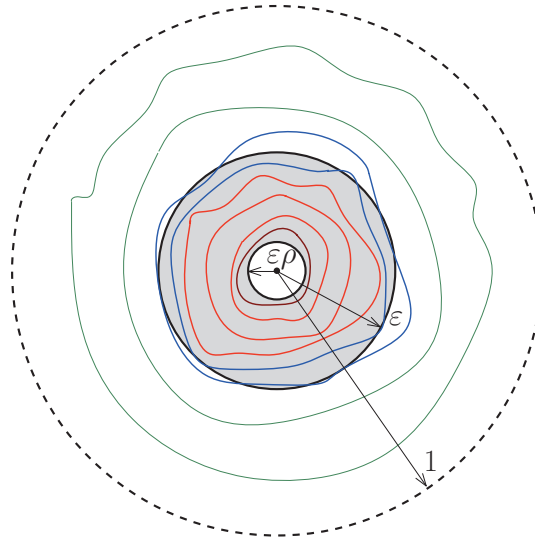
$$\dim_{\mathcal{H}} \Phi_{\mathbf{v}} = 2 - \gamma_{\kappa}(\mathbf{v})$$

$$\gamma_{\kappa}(\mathbf{v}) = \mathbf{v} \Lambda_{\kappa}^*(1/\mathbf{v}), \mathbf{v} \geq 0; \quad \Lambda_{\kappa}^*(x) := \sup_{\lambda \in \mathbb{R}} (\lambda x - \Lambda_{\kappa}(\lambda))$$

$$\Lambda_{\kappa}(\lambda) = \ln \left( \frac{-\cos(4\pi/\kappa)}{\cos \left( \pi \sqrt{\left(1 - \frac{4}{\kappa}\right)^2 + \frac{8\lambda}{\kappa}} \right)} \right)$$

Moment generating function of the loop log-conformal radius *[Cardy & Ziff '02; Kenyon & Wilson '04; Schramm, Sheffield & Wilson '09]*

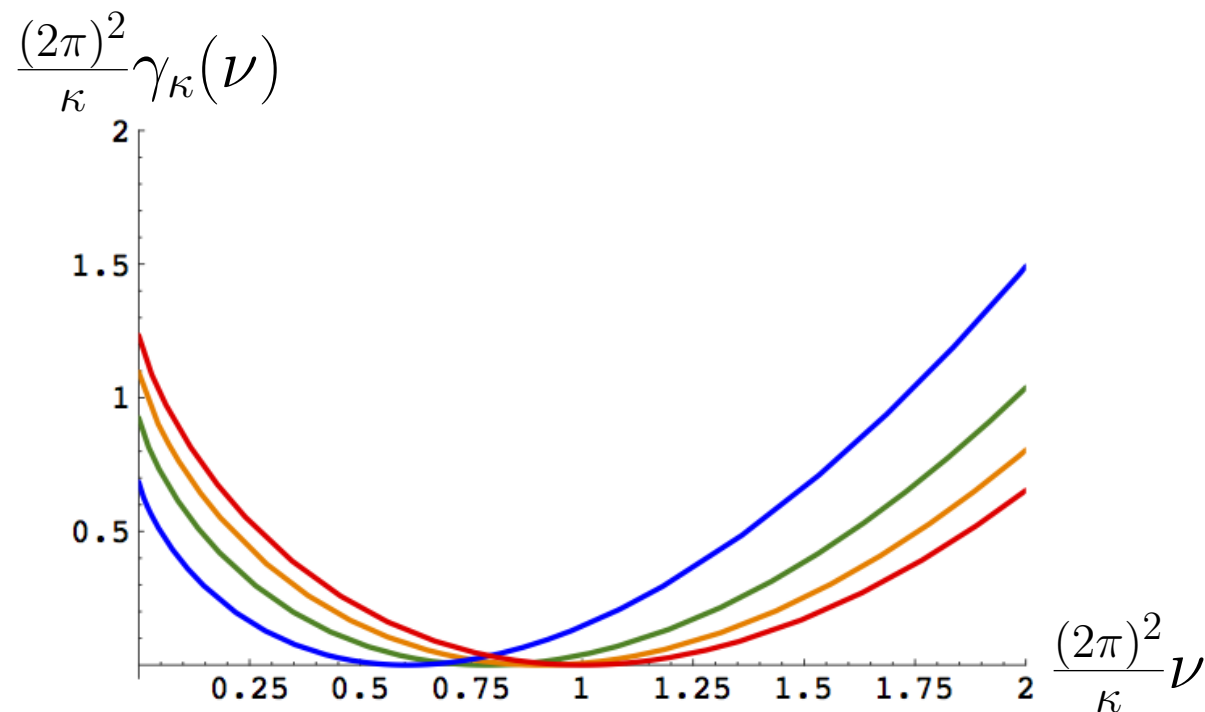
# Conformal Loop Ensemble $\text{CLE}_\kappa$ , $\kappa \in (8/3, 8)$



$\mathcal{U}$  the connected component containing 0 in the complement  $\mathbb{D} \setminus \mathcal{L}$  of the largest loop  $\mathcal{L}$  surrounding 0 in  $\mathbb{D}$ . Cumulant generating function of  $T = -\ln(\text{CR}(0, \mathcal{U}))$  [Schramm, Sheffield, Wilson '09]

$$\Lambda_\kappa(\lambda) := \ln \mathbb{E} \left[ e^{\lambda T} \right] = \ln \left( \frac{-\cos(4\pi/\kappa)}{\cos \left( \pi \left[ \left(1 - \frac{4}{\kappa}\right)^2 + \frac{8\lambda}{\kappa} \right]^{1/2} \right)} \right), \lambda \in \left(-\infty, 1 - \frac{2}{\kappa} - \frac{3\kappa}{32}\right).$$

# Large Deviations Function

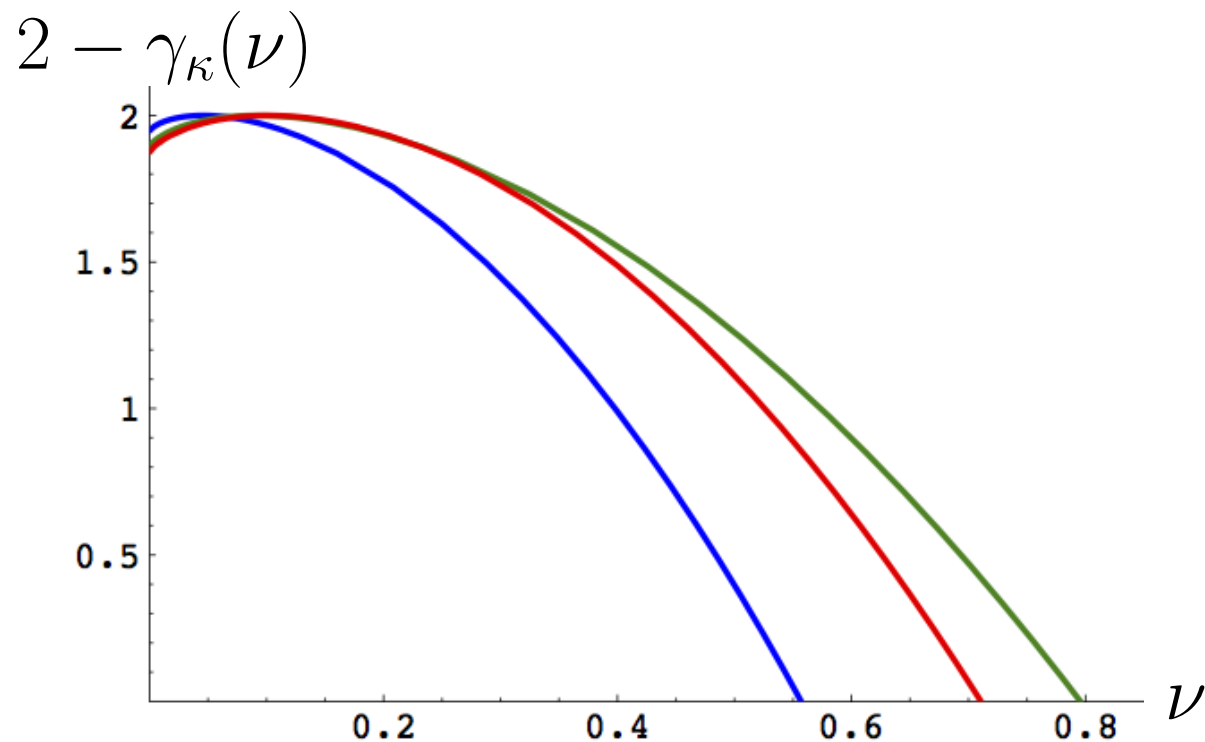


CLE $_{\kappa}$  nesting large deviations function,  $\gamma_{\kappa}(\nu)/\kappa$ ,

for  $\kappa = 3$  or  $6$  (Ising / Percolation,  $n = 1$ ),  $\kappa = 16/3$  (FK-Ising,  $n = \sqrt{2}$ ),

$\kappa = 25/4$  (3-state Potts,  $n = \sqrt{3}$ ),  $\kappa = 4$  (GFF contour lines,  $n = 2$ )

# Multifractal Spectrum



$\text{CLE}_\kappa$  nesting Hausdorff dimension,  $\dim_{\mathcal{H}} \Phi_{\mathbf{v}} = 2 - \gamma_\kappa(\mathbf{v})$ ,  
for  $\kappa = 3$  (Ising),  $\kappa = 4$  (GFF contour lines),  $\kappa = 6$  (Percolation).

## Large Deviations

Euclidean case: for a ball of radius  $\varepsilon$

$$\mathbb{P}(\mathcal{N}_z \approx \mathbf{v} \ln(1/\varepsilon) \mid \varepsilon) = \mathbb{P}(\mathcal{N}_z \approx \mathbf{v}t \mid t) \asymp \varepsilon^{\gamma_{\mathbf{K}}(\mathbf{v})} = \exp[-t\gamma_{\mathbf{K}}(\mathbf{v})].$$

Liouville Quantum Gravity:

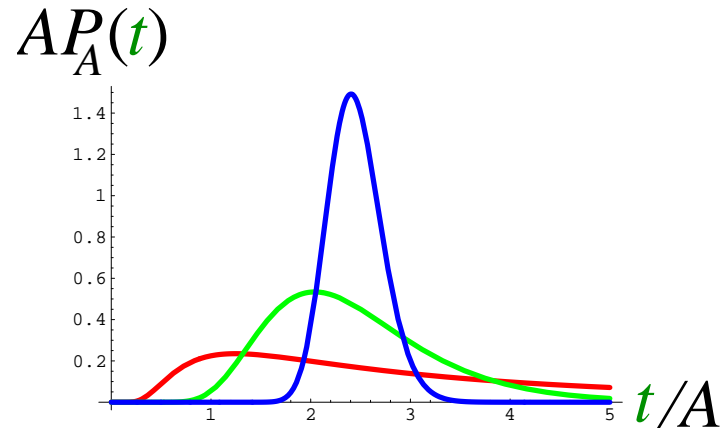
$$t := -\ln \varepsilon; \quad A := -\gamma^{-1} \ln \delta, \quad \delta := \int_{B(z, \varepsilon)} \mu_{\gamma} \quad (\text{quantum ball})$$

Conditioned on  $\delta$ , hence  $A$ , perform the convolution

$$\mathbb{P}_{\mathcal{Q}}(\mathcal{N}_z \mid A) := \int_0^{\infty} \mathbb{P}(\mathcal{N}_z \mid t) P(t \mid A) dt,$$

where  $P(t \mid A)$  is the *probability distribution* of the *random Euclidean log-radius*  $t$ , **given** the quantum log-radius  $A$ .

# Probability Distribution [D.– Sheffield '09]



$$\gamma = \sqrt{8/3} \quad [A = 2; 20; 200]$$

$$P(t | A) = \frac{A}{\sqrt{2\pi t^3}} \exp \left[ -\frac{1}{2t} (A - a_\gamma t)^2 \right]$$

$$t = -\ln \varepsilon, \quad A = -\gamma^{-1} \ln \delta, \quad \delta = \int_{B(z, \varepsilon)} \mu_\gamma$$

$$a_\gamma := 2/\gamma - \gamma/2$$

## Quantum Large Deviations

$$t = -\ln \varepsilon, \quad A = -\gamma^{-1} \log \delta \text{ (quantum ball),}$$

$$\mathcal{N} \approx -\mathbf{v} \ln \varepsilon = \mathbf{v}t, \quad \mathcal{N} \approx -p \ln \delta = \gamma pA,$$

which implies  $\mathbf{v}t = \gamma pA$ . The above convolution then yields, for  $A \rightarrow +\infty$ ,

$$\begin{aligned} \mathbb{P}_Q(\mathcal{N}_z \approx \gamma pA | A) &\asymp \int_0^\infty \frac{dt A}{\sqrt{2\pi t^3}} \exp\left(-\frac{(A - \mathbf{v}t)^2}{2t} - \gamma_{\mathbf{K}}(\mathbf{v})t\right) \\ &\asymp \exp[-A \Theta(p)] \text{ (saddle point at constant } \mathbf{v}t) \end{aligned}$$

$\Theta(p)$  is the large deviations function for the loop number around a  $\delta$ -quantum ball to scale as  $p \log(1/\delta)$ .

## Legendre Transform & KPZ

In the plane, the Legendre transform gave

$$\gamma_{\mathbf{k}}(\mathbf{v}) = \lambda - \mathbf{v}\Lambda_{\mathbf{k}}(\lambda), \quad \frac{1}{\mathbf{v}} = \frac{\partial \Lambda_{\mathbf{k}}(\lambda)}{\partial \lambda}.$$

In Liouville Quantum Gravity

$$\Theta(p) = U_{\gamma}^{-1}(\lambda/2) - p\Lambda_{\mathbf{k}}(\lambda), \quad \frac{1}{p} = \frac{\partial \Lambda_{\mathbf{k}}(\lambda)}{\partial U_{\gamma}^{-1}(\lambda/2)},$$

where  $U_{\gamma}^{-1}(\lambda/2) := (\sqrt{a_{\gamma}^2 + 2\lambda} - a_{\gamma})/\gamma$  is the inverse **KPZ** function, with

$$\gamma = \min \{ \sqrt{\mathbf{k}}, 4/\sqrt{\mathbf{k}} \}, \quad a_{\gamma} = 2/\gamma - \gamma/2.$$





**Theorem** [Borot, Bouttier, D. '16]

In **Liouville quantum gravity**, the cumulant generating function  $\Lambda_{\kappa}$ , with  $\kappa \in (8/3, 8)$ , is transformed into the quantum one,  $\Lambda_{\kappa}^Q := \Lambda_{\kappa} \circ 2U_{\gamma}$ , where  $U_{\gamma}$  is the KPZ function with  $\gamma = \min\{\sqrt{\kappa}, 4/\sqrt{\kappa}\}$ . Its Legendre-Fenchel transform is

$$\Lambda_{\kappa}^{Q*}(x) := \sup_{\lambda \in \mathbb{R}} \left( \lambda x - \Lambda_{\kappa}^Q(\lambda) \right).$$

The quantum nesting distribution in the disk is then, for  $\delta \rightarrow 0$ ,

$$\mathbb{P}_Q(\mathcal{N}_z \approx p \ln(1/\delta) \mid \delta) \asymp \delta^{\Theta(p)},$$

$$\Theta(p) = \begin{cases} p \Lambda_{\kappa}^{Q*}(1/p), & \text{if } p > 0 \\ 3/4 - 2/\kappa & \text{if } p = 0 \text{ and } \kappa \in (8/3, 4] \\ 1/2 - \kappa/16 & \text{if } p = 0 \text{ and } \kappa \in [4, 8). \end{cases}$$

**Corollary** [Borot, Bouttier, D. '16]

The **quantum** generating function associated with  $\text{CLE}_\kappa$  nesting is, for  $\kappa \in (\frac{8}{3}, 8)$

$$\Lambda_\kappa^Q(\lambda) = \Lambda_\kappa \circ 2U_\gamma(\lambda) = \ln \left( \frac{\cos [\pi(1 - 4/\kappa)]}{\cos [\pi(2\lambda/c + |1 - 4/\kappa|)]} \right), \quad c = \max\{1, \kappa/4\},$$

$$\lambda \in \left[ \frac{1}{2} - \frac{2}{\kappa}, \frac{3}{4} - \frac{2}{\kappa} \right] \text{ for } \kappa \in \left( \frac{8}{3}, 4 \right]; \quad \lambda \in \left[ \frac{1}{2} - \frac{\kappa}{8}, \frac{1}{2} - \frac{\kappa}{16} \right] \text{ for } \kappa \in [4, 8).$$

- **The KPZ relation**, which usually concerns scaling dimensions, acts here on a conjugate variable in a Legendre transform.
- **The composition map**  $\Lambda_\kappa \mapsto \Lambda_\kappa^Q = \Lambda_\kappa \circ 2U_\gamma$  to go from Euclidean geometry to Liouville quantum geometry is fairly general.

# CLE Nesting in Liouville Quantum Gravity

**Theorem** [Borot, Bouttier, D. '16]

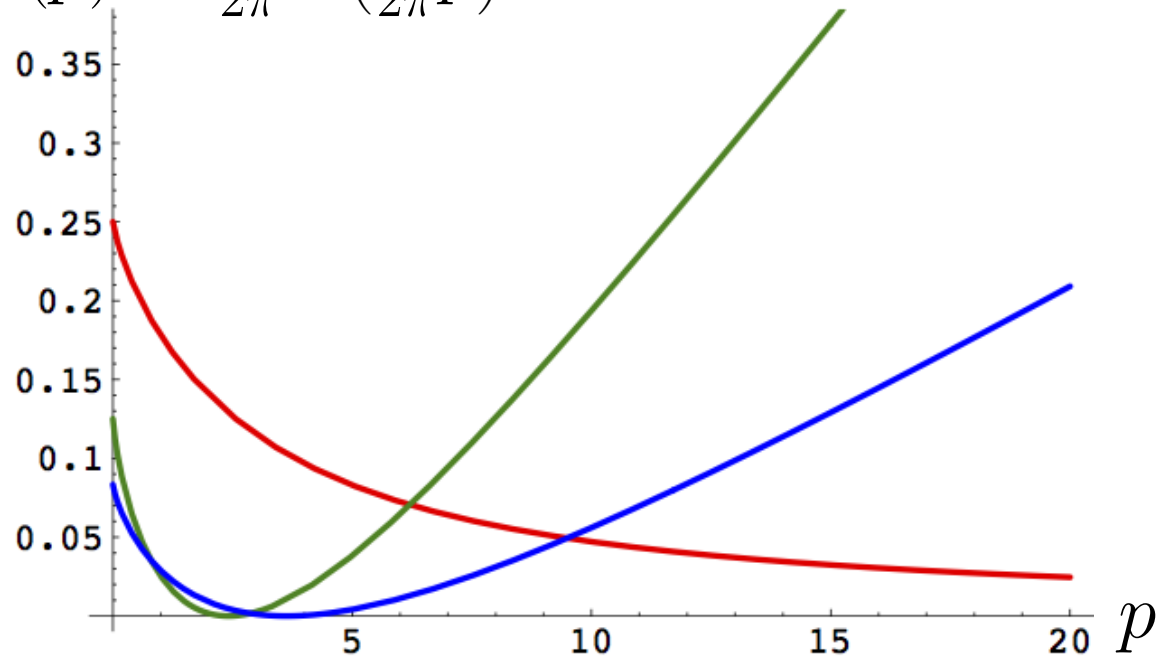
The **quantum** nesting probability of  $\text{CLE}_\kappa$  in a simply connected domain, for the number  $\mathcal{N}_z$  of loops surrounding a ball centered at  $z$  and conditioned to have a given Liouville quantum measure  $\delta$ , has the large deviations form,

$$\mathbb{P}_Q \left( \mathcal{N}_z \approx \frac{cp}{2\pi} \ln(1/\delta) \mid \delta \right) \asymp \delta^{\frac{c}{2\pi} J(p)}, \quad \delta \rightarrow 0,$$
$$\Theta \left( \frac{cp}{2\pi} \right) = \frac{c}{2\pi} J(p),$$

where  $c$  and  $J$  are the *same* as in the **combinatorial result** for the **critical  $O(n)$  model** in the scaling limit of **large random maps**.

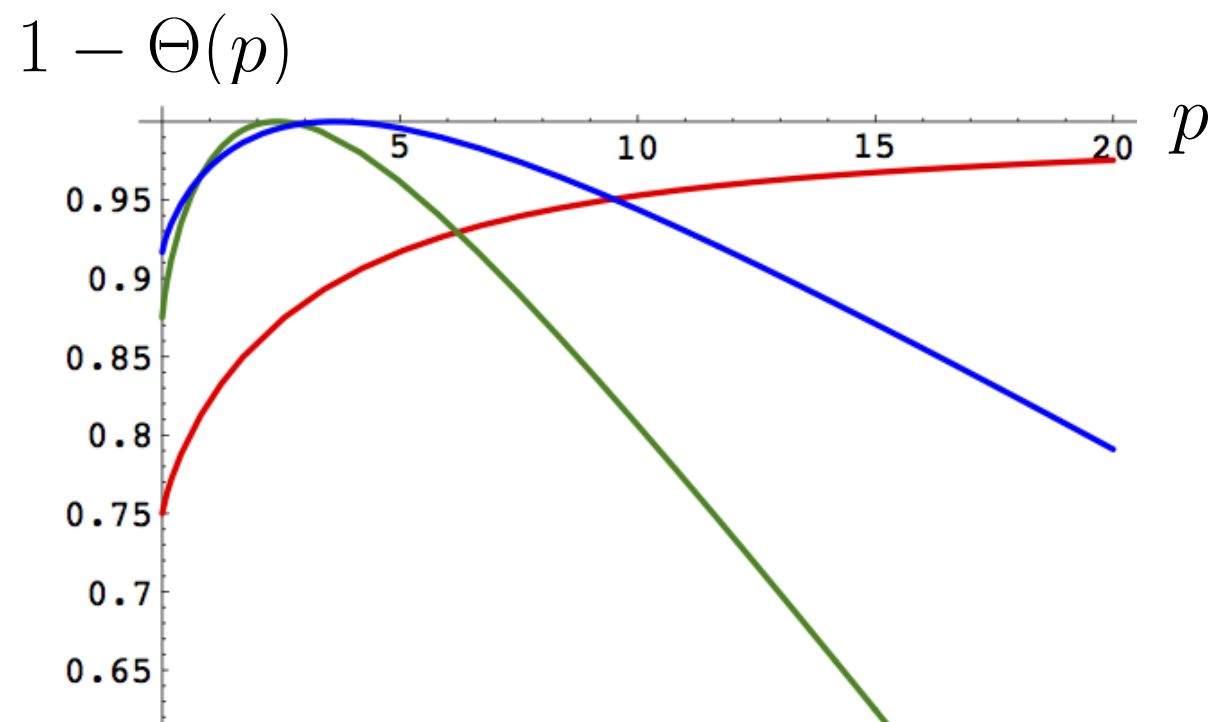
# Quantum Large Deviations Function

$$\Theta(p) = \frac{c}{2\pi} J\left(\frac{c}{2\pi}p\right)$$



$\Theta(p)$  for  $\kappa = 3$  (Ising),  $\kappa = 4$  (GFF contour lines),  $\kappa = 6$  (Percolation).

# Quantum Multifractal Spectrum



$1 - \Theta(p)$  for  $\kappa = 3$  (Ising),  $\kappa = 4$  (GFF contour lines),  $\kappa = 6$  (Percolation).

**Quantum Hausdorff Dimension** for  $p$ -nesting points:  $D_{\mathcal{H}}(1 - \Theta(p))$ ,  
with  $D_{\mathcal{H}}$  Hausdorff dimension of the  $\gamma$ -Liouville quantum surface.

## CLE on the Riemann sphere [Kemppainen & Werner '14]

**Theorem** [Borot, Bouttier, D. '16]

The nesting probability in  $\text{CLE}_\kappa(\widehat{\mathbb{C}})$  between two balls of radius  $\varepsilon_1$  and  $\varepsilon_2$  and centered at two distinct punctures, has the large deviations form,

$$\mathbb{P}^{\widehat{\mathbb{C}}}[\mathcal{N}(\varepsilon_1, \varepsilon_2) \approx \mathbf{v} \ln(1/(\varepsilon_1 \varepsilon_2))] \asymp (\varepsilon_1 \varepsilon_2)^{\gamma_\kappa(\mathbf{v})}, \quad \mathbf{v} \geq 0, \quad \varepsilon_1, \varepsilon_2 \rightarrow 0,$$

where  $\gamma_\kappa(\mathbf{v})$  is the large deviations function of the disk topology.

### Corollary

For two balls of same radius  $\varepsilon$ ,

$$\mathbb{P}^{\widehat{\mathbb{C}}}(\mathcal{N}(\varepsilon, \varepsilon) \approx \mathbf{v} \ln(1/\varepsilon)) \asymp \varepsilon^{\widehat{\gamma}_\kappa(\mathbf{v})}, \quad \mathbf{v} \geq 0, \quad \varepsilon \rightarrow 0,$$

where  $\widehat{\gamma}_\kappa(\mathbf{v})$  is related to the disk large deviations function by

$$\widehat{\gamma}_\kappa(\mathbf{v}) = 2\gamma_\kappa(\mathbf{v}/2).$$

# Quantum Riemann sphere

**Theorem** [Borot, Bouttier, D. '16]

On the quantum sphere  $\widehat{\mathbb{C}}$ , the large deviations function  $\widehat{\Theta}$  which governs the nesting probability between two non-overlapping  $\delta$ -quantum balls,

$$\mathbb{P}_{\widehat{\mathbb{C}}}^{\widehat{\mathbb{C}}}(\mathcal{N} \approx p \ln(1/\delta) \mid \delta) \asymp \delta^{\widehat{\Theta}(p)}, \quad \delta \rightarrow 0,$$

is related to the  $\Theta$  function for the **disk** topology by

$$\widehat{\Theta}(p) = 2\Theta(p/2),$$

so that

$$\mathbb{P}_{\widehat{\mathbb{C}}}^{\widehat{\mathbb{C}}}\left(\mathcal{N} \approx \frac{cp}{\pi} \ln(1/\delta) \mid \delta \rightarrow 0\right) \asymp \delta^{\frac{c}{\pi}J(p)},$$

where  $c$  and  $J$  are the same as before.

**Perfect matching** of **LQG** results for  $\text{CLE}_{\kappa}$  with those for the  $O(n)$  **model** on a **random planar map**, with the correspondence  $\delta \leftrightarrow 1/V$ , with  $\delta \rightarrow 0, V \rightarrow +\infty$ .

HAPPY BIRTHDAY, PETER!

