NUMBER OF COLLISIONS OF BILLIARD BALLS

Krzysztof Burdzy
University of Washington

Collaborators and advisers

Collaborator: Jayadev Athreya and Mauricio Duarte

Work in progress.

We are most grateful to (in chronological order): Jaime San Martin, Curt McMullen, Branko Grünbaum, Rekha Thomas.

Billiard balls

Consider n billiard balls ($n < \infty$) in a d-dimensional Euclidean space. The billiards table has no boundary.

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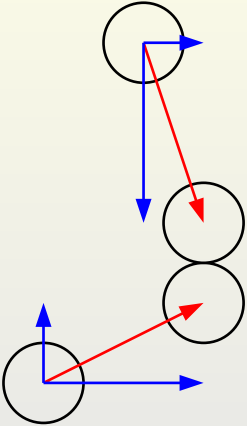
The balls have equal radii and equal masses.

The collisions between the balls are totally elastic. The total energy, total momentum and total angular momentum are preserved.

Two balls



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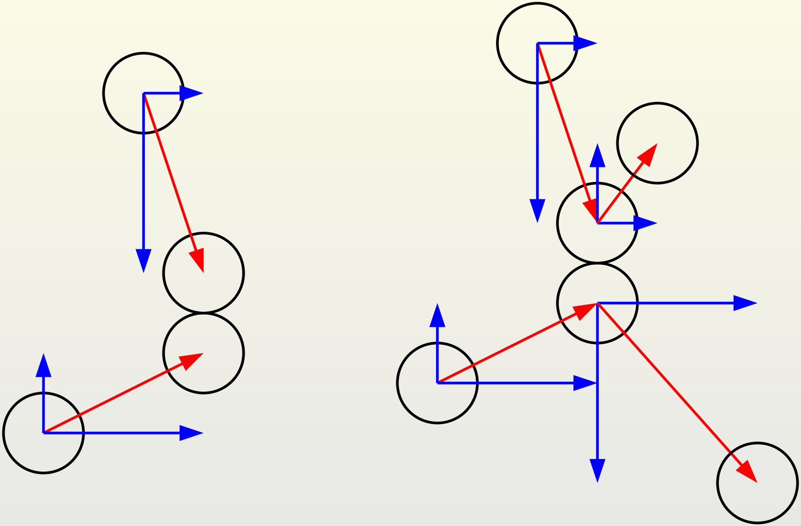
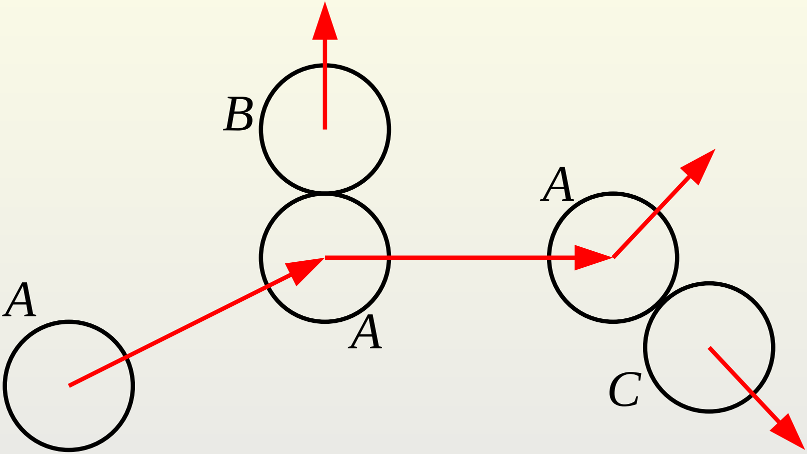


Two balls: number of collisions

Two balls may have zero collisions or one collision.

Three balls



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Let $K(n, d)$ be the supremum of the number of collisions of n balls in d-dimensional space. The supremum is taken over all initial conditions (positions and velocity vectors).

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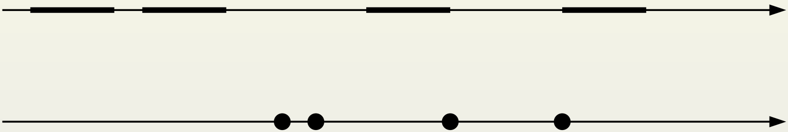
Can three balls have more than two collisions?

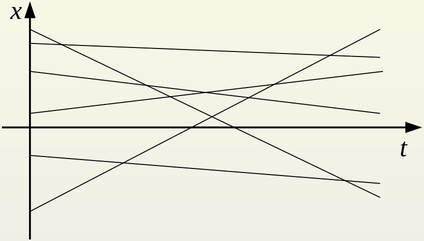
Let $K(n, d)$ be the supremum of the number of collisions of n balls in d-dimensional space. The supremum is taken over all initial conditions (positions and velocity vectors).

Possible values of $K(3, 2)$ (the supremum of the number of collisions for three balls in the plane):
$2, 3, 4, 5, 6, \ldots$
∞ — not attained
∞ — attained

The evolution of one dimensional balls (rods) is equivalent to the evolution of reflecting point masses. The evolutions of the gaps are identical.



A graphical representation of one dimensional evolution



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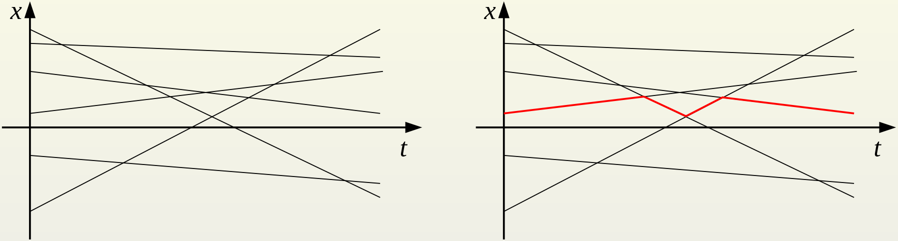

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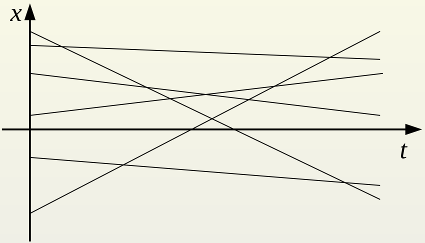
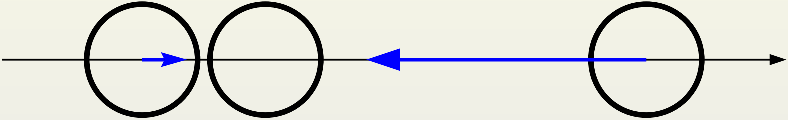


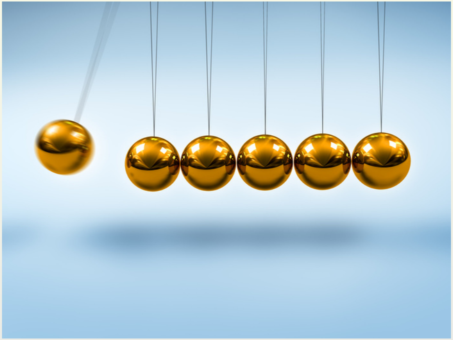
The number of intersections of n half-lines can be any integer in the range from 0 to $n(n-1)/2$ (= the number of pairs of distinct half-lines).

Aligned balls in d dimensions

The one dimensional analysis applies to n balls with centers on a single line in d dimensions, assuming that the velocity vectors are parallel to this line.



Beads on a line



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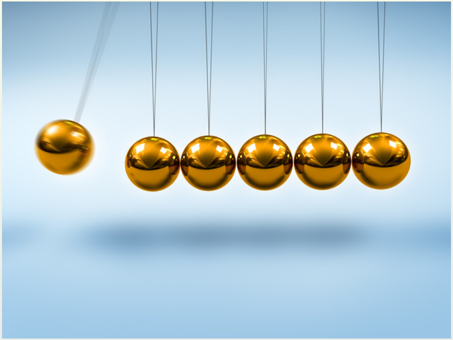

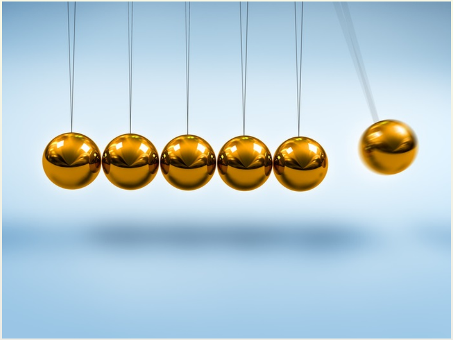

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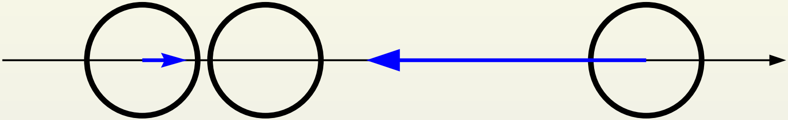

The number of collisions of three balls moving on a line can be any of the numbers $0, 1, 2, 3 = 3(3-1)/2$.

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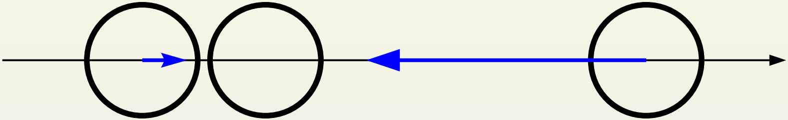

The number of collisions of three balls moving on a line can be any of the numbers $0, 1, 2, 3 = 3(3-1)/2$.

Can the number of collisions of three balls in the plane be larger than 3?

Historical remarks

Murphy and Cohen (2000): In the 60's, Uhlenbeck conjectured that the maximum number of collisions of three balls in any number of dimensions was 3. Intuition suggests that the balls are most "constrained" in one dimension.

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Sinai asked whether a finite family of balls can have an infinite number of collisions. (70's ?)

Upper bounds

THEOREM (Vaserstein (1979))

For any number of balls, dimension of the space and initial conditions, the number of collisions is finite.

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THEOREM (Burago, Ferleger and Kononenko (1998))

$$K(n,d) \leq \left(32 n^{3/2}\right)^{n^2},$$

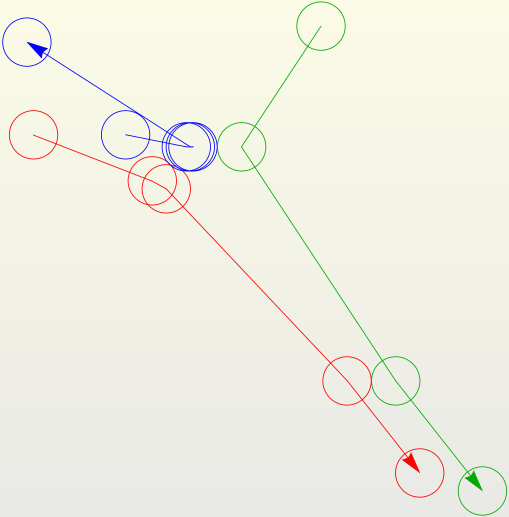
$$K(n,d) \leq \left(400 n^2\right)^{2n^4}.$$

Lower bounds

EXAMPLE (Foch (1960's, unpublished), Murphy and Cohen (2000))

$$K(3,2) \geq 4 > 3(3-1)/2 = K(3,1).$$

Foch's example



Three balls — the answer

THEOREM (Cohen (1966), Murphy and Cohen (1993))

$$K(3, d) \leq 4.$$

Therefore, for $d \geq 2$,

$$K(3, d) = 4 > 3(3-1)/2 = K(3, 1).$$

THEOREM (Cohen (1966), Murphy and Cohen (1993))

$$K(3,d) \le 4.$$

Therefore, for $d \ge 2$,

$$K(3,d) = 4 > 3(3-1)/2 = K(3,1).$$

There are no theorems (examples) in the existing literature showing that $K(n,d) > K(n,1)$ for any $n \ge 4$, $d \ge 2$.

New lower bound

THEOREM (B and Duarte (2017))

$$K(n,d) \geq K(n,2) \geq f(n) > n^3/27, \qquad d \geq 2.$$

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$$f(n) > n(n-1)/2 = K(n,1) \qquad \text{for } n \geq 7,$$
$$f(n) = n(n-1)/2 = K(n,1) \qquad \text{for } n = 6,$$
$$f(n) < n(n-1)/2 = K(n,1) \qquad \text{for } 3 \leq n \leq 5.$$

Improvement of Foch's example

THEOREM (B and Duarte (2017))

For $n \geq 3, d \geq 2$,

$$K(n,d) \geq K(n,2) \geq 1 + n(n-1)/2 > n(n-1)/2 = K(n,1).$$

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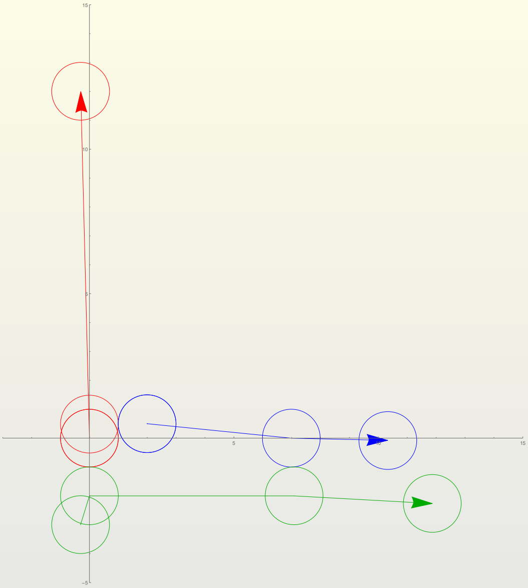
OPEN PROBLEM

Do there exist $n \geq 2$ and $d_1 > d_2 \geq 2$ such that

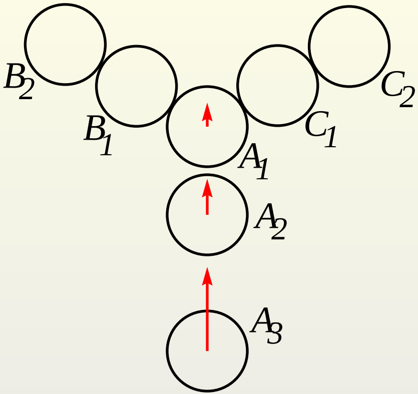
$$K(n,d_1) > K(n,d_2)?$$

The answer is negative for $n = 2, 3$.

Conceptual version of Foch's example

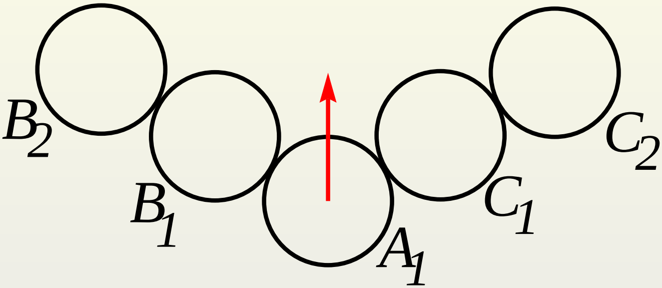


Main example



$$K(n,2) > n^3/27.$$

Evolution (1)



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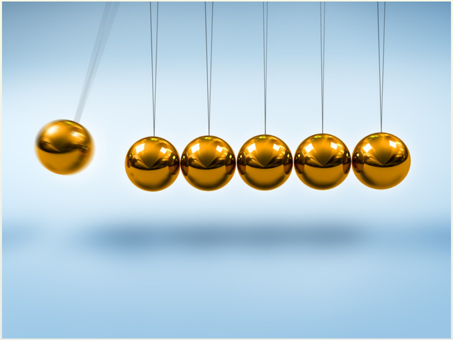
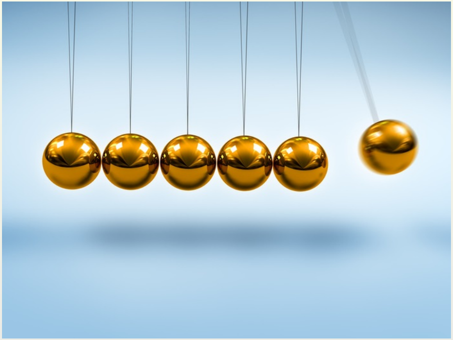
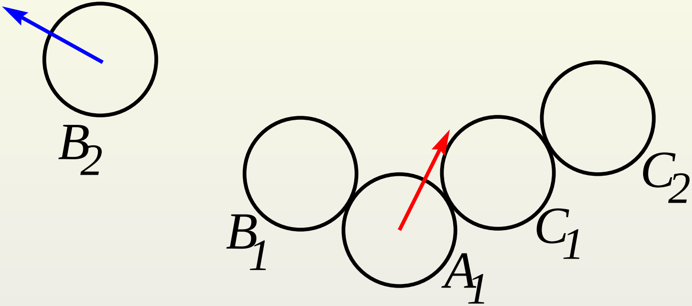
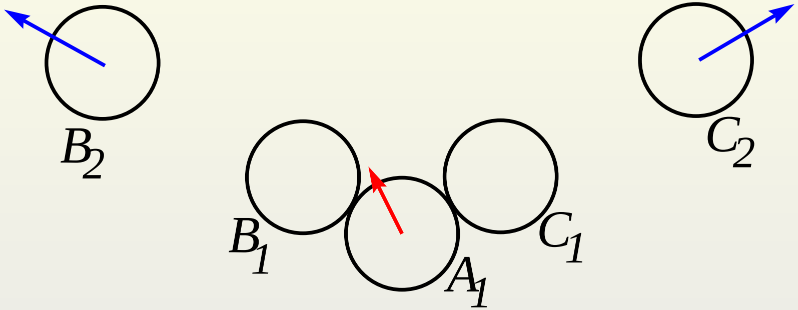




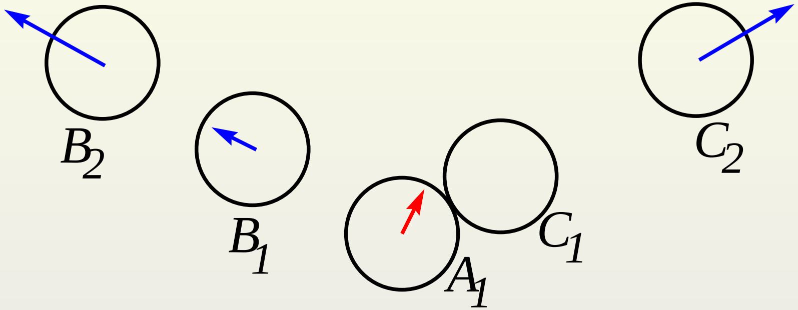
Evolution (2)



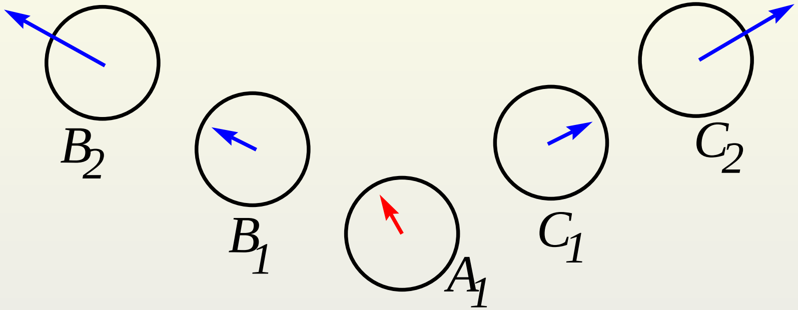
Evolution (3)



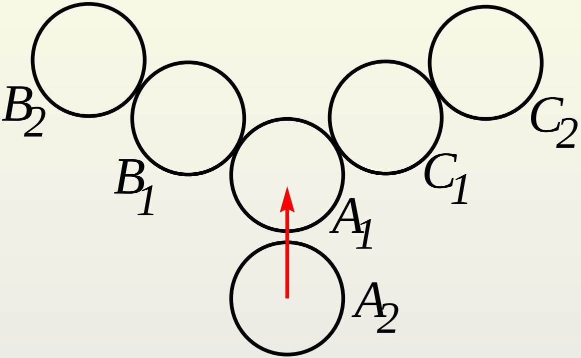
Evolution (4)



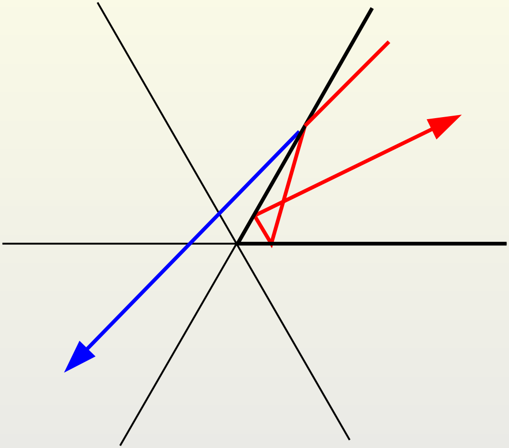
Evolution (5)



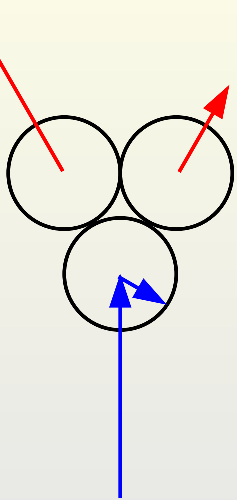
Evolution (6)



Reflections near corner



Multiple collisions



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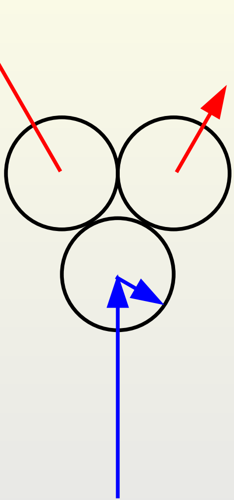
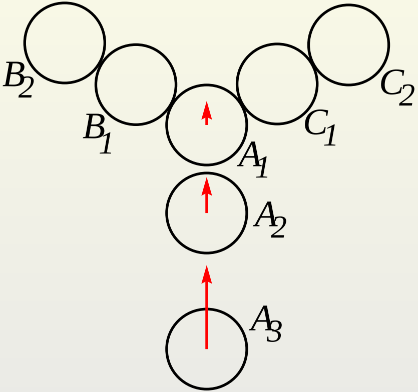
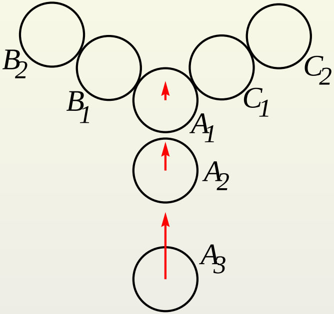




Alternative configuration



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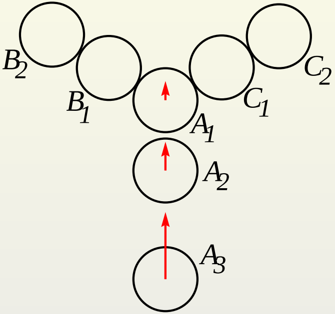
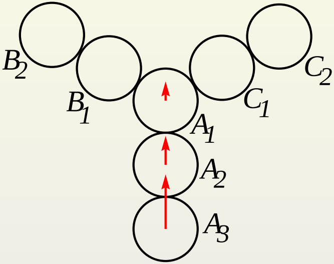
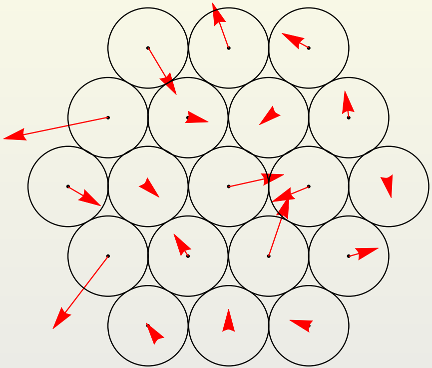




The initial configuration can be arbitrarily tight.

Pinned billiard balls



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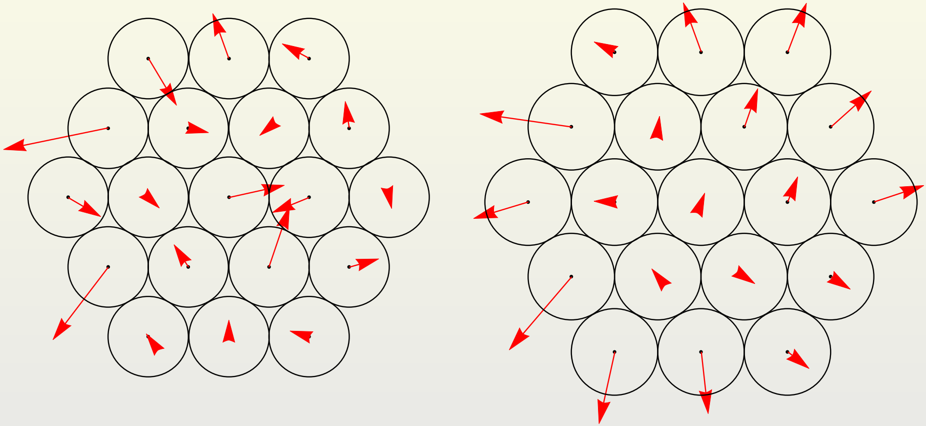
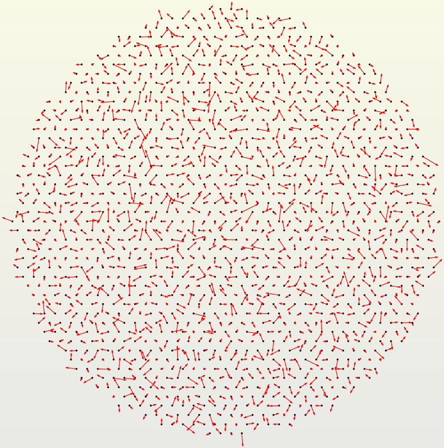


Large family of pinned balls



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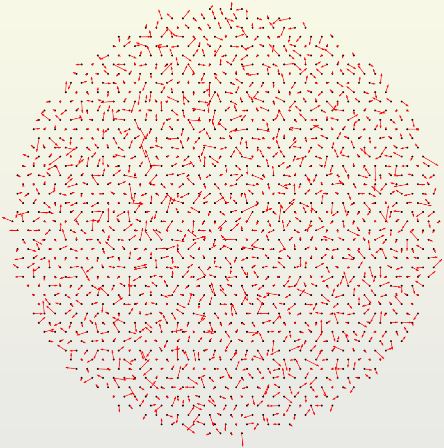

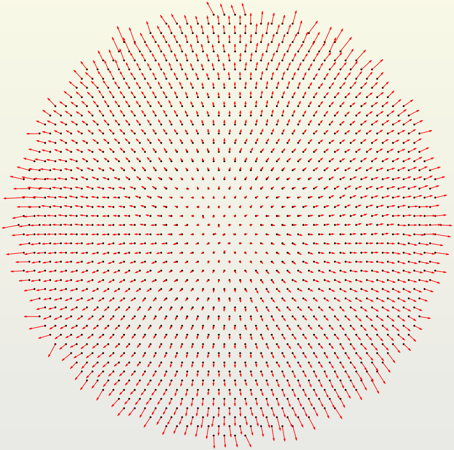

Finite evolution

Consider n balls which may touch but not overlap. The balls are labeled 1 to n. The balls are pinned (they cannot move).
The k-th ball is associated with a pseudo-velocity vector v_k.

Consider an infinite sequence of pairs of labels $\{(i_k, j_k), k \geq 1\}$. We have $1 \leq i_k, j_k \leq n$. The pairs of balls undergo pseudo-collisions, in the order determined by the sequence. If $i_k = j_k$ or the balls labeled i_k and j_k do not touch then v_{i_k} and v_{j_k} remain unchanged. If the balls touch then the pseudo-velocities change according to the law of elastic collision.

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PROPOSITION (Athreya, B and Duarte)

For any family of balls, pseudo-velocities v_k and sequence $\{(i_k, j_k), k \geq 1\}$, the pseudo-velocities will freeze after a finite number of pseudo-collisions.

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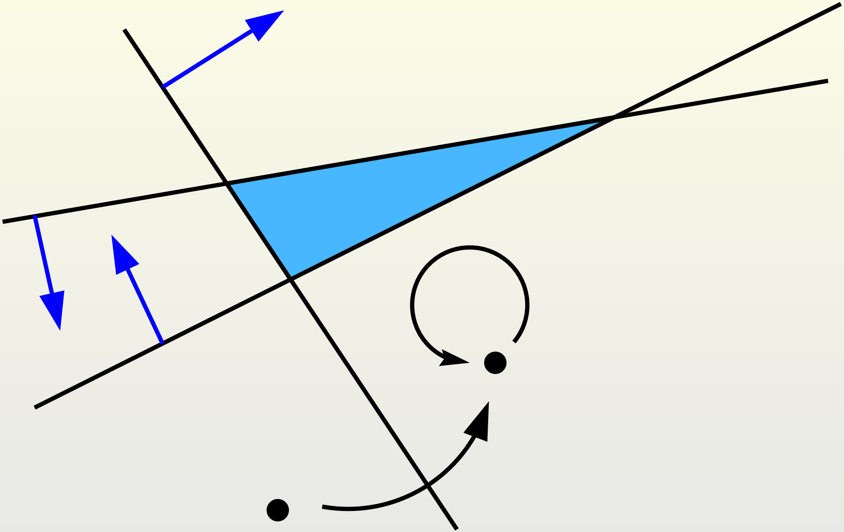
OPEN PROBLEM

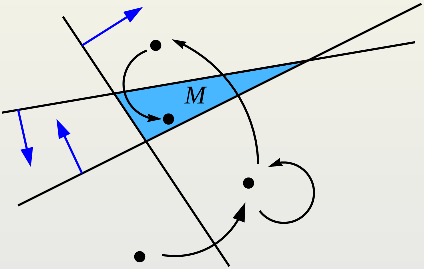
Can the above proposition be derived from the theorems of Vaserstein or Burago, Ferleger and Kononenko?

Foldings



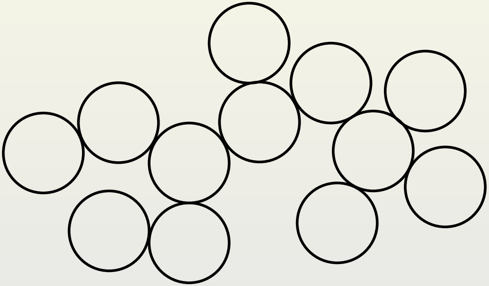
PROPOSITION (Athreya, B and Duarte)

Suppose that a family of open halfspaces passing through 0 has a non-empty intersection M. Consider a point x and a sequence $F_1, F_2, \dots$ of foldings relative to these halfspaces such that every halfspace is represented infinitely often. Then there exists k such that $F_k \circ F_{k-1} \circ \cdots \circ F_2 \circ F_1(x) \in M$.



CONJECTURE (Athreya, B and Duarte)

For some $c_1, c_2 < \infty$, any tree-like family of pinned balls, any initial pseudo-velocities and any sequence $\{(i_k, j_k), k \geq 1\}$ of pseudo-collisions, the pseudo-velocities will freeze after at most $c_1 n^{c_2 n}$ pseudo-collisions.



Why is our upper bound so large?

Our proof contains an inductive argument which gives a bound of the following type:

$$K(n,d) \approx nK(n-1,d).$$

This implies that

$$K(n,d) \approx n! \approx n^n.$$

Distant collisions

Consider a family of n balls in d dimensional space.
The center of the k-th ball: $x_k(t)$; the velocity of the k-th ball: $v_k(t)$.
$\mathbf{x}(t) = (x_1(t), \dots, x_n(t)) \in \mathbb{R}^{dn}$, $\qquad \mathbf{v}(t) = (v_1(t), \dots, v_n(t)) \in \mathbb{R}^{dn}$

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(i) The family of n balls can be partitioned into two non-empty subfamilies such that no ball from the first family collides with a ball in the second family after time $500n^3|\mathbf{x}(0)|$.

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(ii) The family of n balls can be partitioned into two non-empty subfamilies such that no ball from the first family collides with a ball in the second family when $|\mathbf{x}(t)| \geq (1 + 500n^3)|\mathbf{x}(0)|$.

An upper bound for the number of collisions

COROLLARY/CONJECTURE (B and Duarte)

Suppose that for every collision between two balls, the distance between any other pair of balls is greater than n^{-n}. Then the total number of collisions is bounded by $c_1 n^{c_2 n}$.

A monotone functional

Consider a family of n balls in d dimensional space.

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Let $\widetilde{\mathbf{x}}(t)$ and $\widetilde{\mathbf{v}}(t)$ have the analogous meaning for the system of non-interacting balls with the same initial conditions.

A monotone functional

Consider a family of n balls in d dimensional space.
The center of the k-th ball: $x_k(t)$; the velocity of the k-th ball: $v_k(t)$.
$\mathbf{x}(t) = (x_1(t), \dots, x_n(t)) \in \mathbb{R}^{dn}$, $\qquad \mathbf{v}(t) = (v_1(t), \dots, v_n(t)) \in \mathbb{R}^{dn}$

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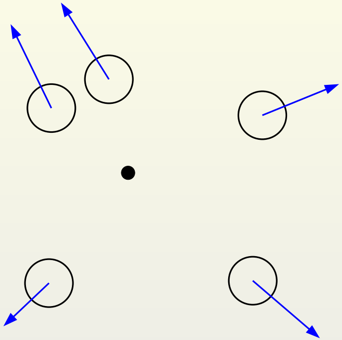
$$\frac{\mathbf{x}(t)}{|\mathbf{x}(t)|} \cdot \mathbf{v}(t) \geq \frac{\widetilde{\mathbf{x}}(t)}{|\widetilde{\mathbf{x}}(t)|} \cdot \widetilde{\mathbf{v}}(t).$$

The end of supernova explosion



Vaserstein (1979), Illner (1990): qualitative arguments; the number of collisions is finite.
B and Duarte (2017): quantitative arguments; an upper bound for the distance of collision locations.

Time reversibility

"New makers of modern culture" by Justin Wintle, editor, (2007), article on Ludwig Boltzmann by Heinz Post, page 174:

> *A certain function, called entropy, of the amount of heat transferred divided by the temperature, can only increase. On the other hand, mechanics, to which Boltzmann wanted to reduce thermodynamics, is strictly reversible, in the sense that for every motion there is another motion described by reversing the sign of the variable denoting time, which is equally possible. The opponents of Boltzmann's atomism kept pointing out that you cannot expect to obtain irreversibility from a theory according to which all processes are essentially reversible.*

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Entropy is monotone.

Hence, the universe is not made of atoms.

Velocities and speeds

The center of the k-th ball: $x_k(t)$.
The velocity of the k-th ball: $v_k(t)$.
The speed of the k-th ball: $|v_k(t)|$.

Non-existence of atoms revisited

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FALSE: Therefore, there are no (non-constant) monotone functions of billiard ball positions and **velocities**.

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Is entropy an increasing function of billiard ball positions and speeds?

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DEFINITIONS
(i) Physics: Entropy is the logarithm of the number of (equally probable) microscopic configurations corresponding to the given macroscopic state of the system.
(ii) Applied mathematics (KB): Physical entropy is the mathematical (probabilistic) entropy of the empirical distribution of positions and momenta for the coarse grained model of the system.