Quasisymmetric rigidity for Sierpiński carpets

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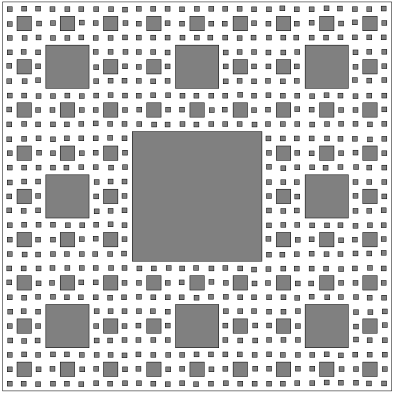
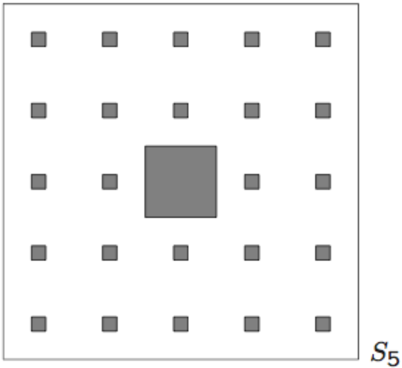
The standard Sierpiński carpet S_3



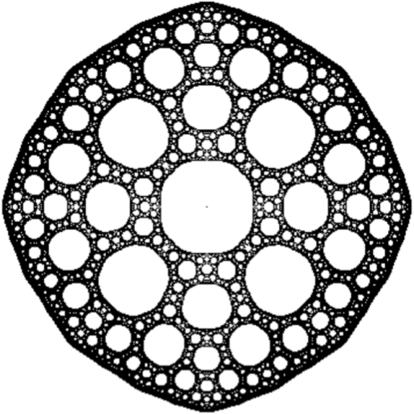
Carpet: Metric space homeomorphic to the standard Sierpiński carpet.

Standard square carpets

The standard square Sierpiński carpet S_p, p odd, is defined as follows: Subdivide the unit square into $p \times p$ squares of equal size, remove the middle, repeat on the remaining squares, etc.

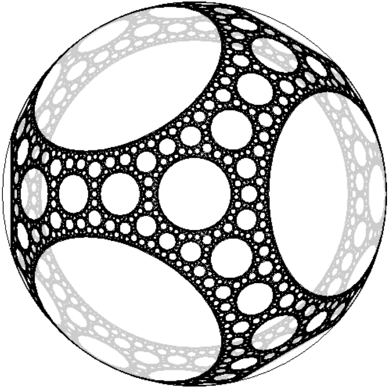


Sierpiński carpets can be Julia sets



The Julia set of the function $f(z) = z^2 - \dfrac{1}{16z^2}$.

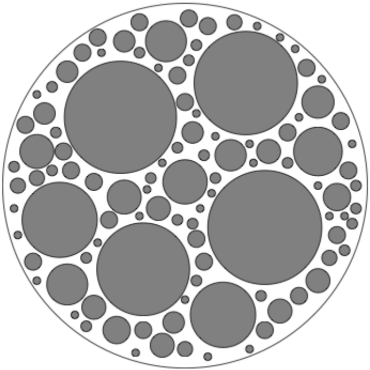
Sierpiński carpet as a limit set of a Kleinian group



Limit set of a (convex cocompact) Kleinian group acting on $\mathbb{H}^3$

Round carpets

A *round carpet* is a carpet embedded in the Riemann sphere $\widehat{\mathbb{C}}$ whose peripheral circles are geometric circles.



Möbius transformations preserve the class of round carpets.

Topological properties of carpets

Whyburn (1958):

- A metric space S is a carpet if and only if it is a planar continuum of topological dimension one, is locally connected, and has no local cut points.
- If $S = \widehat{\mathbb{C}} \setminus \bigcup D_i$, where D_i are pairwise disjoint open Jordan regions for $i \in \mathbb{N}$, then S is a carpet if and only if
  - S has empty interior,
  - $\partial D_i \cap \partial D_j = \emptyset$ for $i \neq j$,
  - $\operatorname{diam}(D_i) \to 0$.

The Kapovich-Kleiner conjecture

Version I

Suppose G is a Gromov hyperbolic group s.t. $\partial_\infty G$ is a carpet. Then G admits a discrete, cocompact, and isometric action on a convex subset of $\mathbb{H}^3$ with non-empty totally geodesic boundary.

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This is equivalent to:

Version II

Suppose G is a Gromov hyperbolic group s.t. $\partial_\infty G$ is a carpet. Then there exists a quasisymmetric homeomorphism of $\partial_\infty G$ onto a round carpet in $\widehat{\mathbb{C}}$.

Inradius and outradius of images of balls

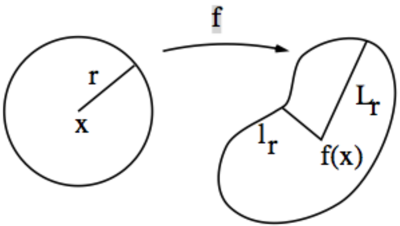
Let (X, d_X) and (Y, d_Y) be metric spaces, and $f\colon X \to Y$ be a homeomorphism.
Define

$$L_r(x) := \sup\{d_Y(f(z), f(x)) : z \in B(x, r)\},$$

and

$$l_r(x) := \inf\{d_Y(f(z), f(x)) : z \in X \setminus B(x, r)\}.$$

$l_r(x)$ is the "inradius" and $L_r(x)$ the "outradius" of the image $f(B(x, r))$ of the ball $B(x, r)$.



Classes of homeomorphisms

The homeomorphism $f\colon X \to Y$ is called:

- *conformal* if $\limsup_{r\to 0} \dfrac{L_r(x)}{l_r(x)} = 1$ for all $x \in X$,
- *quasiconformal (=qc)* if there exists a constant $H \geq 1$ such that $\limsup_{r\to 0} \dfrac{L_r(x)}{l_r(x)} \leq H$ for all $x \in X$,
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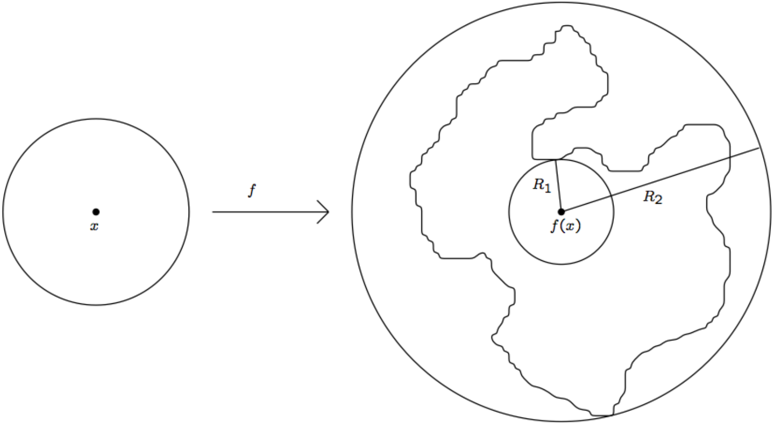
Geometry of a quasisymmetric map



$$R_2/R_1 \leq \text{Const.}$$

Remarks

- f is quasisymmetric if it maps balls to "roundish" sets of uniformly controlled eccentricity.
- Quasisymmetry is on the one hand *weaker* than conformality, because we allow distortion of small balls; on the other hand it is *stronger*, because we control distortion for *all* balls.
- bi-Lipschitz $\Rightarrow$ qs $\Rightarrow$ qc.
- For homeos on $\mathbb{R}^n$, $n \geq 2$: qs $\Leftrightarrow$ qc.

Definition. Two metric spaces X and Y are *qs-equivalent* if there exists a quasisymmetric homeomorphism $f : X \to Y$.

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The quasisymmetric Riemann mapping theorem

Theorem (Ahlfors 1963)

A region $\Omega \subseteq \mathbb{C}$ is qs-equivalent to $\mathbb{D}$ if and only if Ω is a Jordan domain bounded by a quasicircle.

Definition. A Jordan curve $J \subseteq \mathbb{C}$ is called a *quasicircle* iff it is qs-equivalent to the unit circle $\partial\mathbb{D}$.
This is true if and only if there exists a constant $K \geq 1$ such that

$$\operatorname{diam}(\gamma) \leq K|x - y|,$$

whenever $x, y \in J$, and γ is the smaller subarc of J with endpoints x and y.

Qs-equivalence of carpets

Basic Problem. When are two carpets X and Y qs-equivalent?

Theorem (B., Kleiner, Merenkov 2005)

Every quasisymmetry between two round carpets of measure 0 is a Möbius transformation.

Corollary

Two round carpets of measure 0 are qs-equivalent if and only if they are Möbius equivalent.

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The set of qs-equivalence classes of round carpets has the cardinality of the continuum.

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Outline for the proof of the theorem

Let $S, S' \subseteq \widehat{\mathbb{C}}$ be round carpets with $|S| = 0$ and $\varphi\colon S \to S'$ be a quasisymmetry.

1. Extend φ to a quasiconformal map $\varphi\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by successive reflections.

2. For a.e. $z \in \widehat{\mathbb{C}}$: (i) z does not lie in any of the countably many copies of S obtained by reflection and (ii) the linear map $D\varphi(z)$ is non-singular.

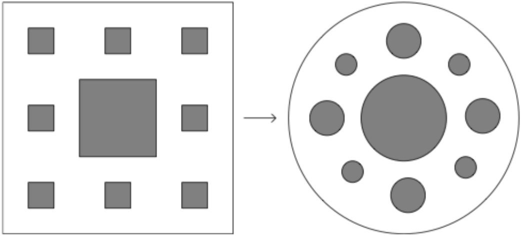
3. For such z: there is a sequence of (geometric) disks D_i with $\mathrm{diam}(D_i) \to 0$ such that $z \in D_i$ and $\varphi(D_i)$ is a disk.
Then $D\varphi(z)$ maps some disk to a disk and so $D\varphi(z)$ is conformal.

4. φ is 1-quasiconformal and hence a Möbius transformation. □

Uniformization

Theorem (B. 2004)

Let $S \subseteq \widehat{\mathbb{C}}$ be a carpet whose peripheral circles are uniform quasicircles *with* uniform relative separation. *Then S is qs-equivalent to a round carpet.*



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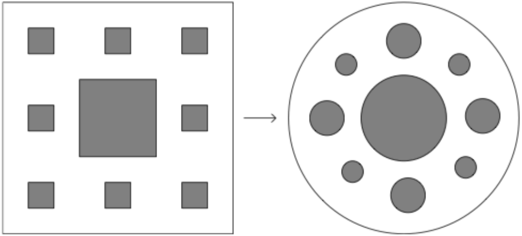

Geometric properties of peripheral circles

- They are *uniform quasicircles* if they satisfy the quasicircle condition with the same parameter K.
- They have *uniform relative separation* if there exists a constant $\delta > 0$ such that

$$\frac{\operatorname{dist}(C, C')}{\min\{\operatorname{diam}(C), \operatorname{diam}(C')\}} \geq \delta > 0,$$

  whenever C and C' are two distinct peripheral circles.

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True for: standard carpets $S_3, S_5, \ldots$, carpets that arise as Julia sets of subhyperbolic rational maps, round group carpets arising as limit sets of Kleinian groups, carpets that are boundaries of Gromov hyperbolic groups.

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Definition. A carpet is called *geometric* if it has peripheral circles that are uniform quasicircles and have uniform relative separation.

The group $QS(X)$ of quasisymmetries

Definition. Let X be a metric space. Then we define

$$QS(X) := \{\varphi\colon X \to X \text{ is a quasisymmetry}\}.$$

- $QS(X)$ is a group.
- If X and Y are qs-equivalent, then $QS(X)$ and $QS(Y)$ are isomorphic.
- For carpets $\partial_\infty G$ the group $QS(\partial_\infty G)$ is large: it is countably infinite (its action on $\partial_\infty G$ is cocompact on triples).
- There are geometric carpets X for which $QS(X)$ is uncountable (Merenkov's slit carpets).
- If $X \subseteq \widehat{\mathbb{C}}$ is a geometric carpet with $|X| = 0$, then $QS(X)$ is countable (it is isomorphic to a discrete group of Möbius transformations).

Rigidity for square carpets

Theorem (B., Merenkov 2005, 2013)

Every quasisymmetry $\varphi\colon S_p \to S_p$, $p \geq 3$ odd, is an isometry, i.e., one of the obvious reflections or rotations that preserve S_p.

Corollary

$QS(S_p)$ is finite, and so no S_p is qs-equivalent to a group carpet.

Theorem (B., Merenkov 2005)

Let p, q be odd numbers. Then S_p and S_q are qs-equivalent if and only if $p = q$.

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Rigidity for Julia set carpets

Theorem (B., Lyubich, Merenkov 2016)

Let $f, g \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be postcritically-finite rational maps whose Julia sets J_f and J_g are carpets. If $\varphi \colon J_f \to J_g$ is a quasisymmetry, then φ is (the restriction of) a Möbius transformation.

Theorem (B., Lyubich, Merenkov 2016)

Let f be postcritically-finite rational map whose Julia set J_f is a carpet. Then $QS(J_f)$ is a finite group of (restrictions of) Möbius transformations.

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Ideas for the proof

1. One would like to relate f, g, and φ. Candidate relation $\varphi \circ f^k = g^n \circ \varphi$ not true in general, but a relation of the form

$$g^m \circ \varphi \circ f^k = g^{m+n} \circ \varphi \quad \text{on } \mathcal{J}_f.$$

This uses uniformization and recent deep rigidity results by S. Merenkov on "relative Schottky sets".

2. One uses this to extend φ to a quasisymmetry on $\widehat{\mathbb{C}}$ so that φ is conformal on the Fatou components of f.

3. Then φ is 1-quasiconformal on $\widehat{\mathbb{C}}$ and hence a Möbius transformation.

Square carpets and Julia set carpets

Theorem (B., Merenkov 2014)

No standard square carpet S_p, p odd, is qs-equivalent to a carpet J_f arising as the Julia set of a postcritically-finite rational map f.

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Open problem

What are the quasisymmetries of the standard Menger sponge?