DUAL FIBRATION OF A PROJECTIVE LAGRANGIAN FIBRATION

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Abstract. We give a definition of dual fibration of an abelian fibration and we show that the dual fibration of a Lagrangian fibration is also Lagrangian with respect to a natural symplectic form.

Introduction

Mukai observed in his paper [15] that the moduli space of sheaves on a symplectic algebraic surface, i.e., an abelian surface or a K3 surface, has a natural symplectic form. If our surface is a K3 surface and the resulted moduli space is 2-dimensional, then the moduli space is also a K3 surface. Moreover, it is known that these two K3 surfaces have equivalent derived categories. This phenomenon has been deeply studied by several authors partly in connection to the homological mirror symmetry conjecture or as a duality problem in the classification theory of algebraic varieties.

In this way it is expected that a geometric object and the moduli space of the sheaves on it are, in general, strongly related in some sense. If we have a fiber space $f : X \to S$, the moduli space of sheaves which are flat with respect to f is particularly interesting when we consider the geometry of f. In particular, the Picard space, i.e., the moduli space of f-line bundles, roughly speaking, is one of the most basic and accessible objects with an analogy to the Jacobian of a curve.

In this paper we define the *dual fibration* of an abelian fibration as such a moduli space following the idea of Raynaud [18] and study the case where the original f is a Lagrangian fibration. More precisely, we define the dual fibration as a connected component of the biggest separated quotient of the Picard space associated to the original abelian fibration (cf. Definition 1.3). We say that f has analytic local sections if for any point $s \in S$ we have an Euclid neighborhood V_s such that the restricted fiber space $f_{|V_s}$ has a section. Our main result is the following:

Main Theorem (=Theorem 5.1)**.** *Let X be a symplectic Kähler manifold and $f : X \to S$ a projective Lagrangian fibration which has analytic local sections and assume that $R^1 f_* \mathcal{O}_X$ is locally free. Let $\pi : Q \to S$ be the dual fibration of f. Then there exists a natural symplectic form on Q. Moreover, the dual fibration*

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$\pi : Q \to S$ is also a (non-proper, in general) Lagrangian fibration with respect to this symplectic form.

A compact symplectic Kähler manifold is said to be *irreducible* if it is simply connected and the space of holomorphic 2-forms is spanned by the symplectic form on it (cf. Definition 1.1, (iii)). If the total space X of a Lagrangian fibration $f : X \to S$ is projective irreducible symplectic manifold and the base S is also projective, the condition of locally freeness of the higher direct image sheaf $R^1 f_* \mathcal{O}_X$ in the Main Theorem is automatically satisfied under the existence of analytic local sections, due to a theorem of Matsushita [14].

We must note that the dual fibration which we utilize here is *not proper*. By making this choice, we can handle Lagrangian fibrations with reducible fibers. However our dual fibration itself is insufficient as an object of studies in algebraic geometry. Therefore it will be necessary to seek some compactification of our dual fibration. It is expected that a compactification is given, in some sense, by some moduli space of semi-stable sheaves supported on the fibers of the original fibration. But an analysis of the moduli space will be related to an involved study of torsion free sheaves on degenerate fibers and the theory of symplectic singularities or the birational geometry on irreducible symplectic manifolds. Since our dual fibration has good properties such as the canonical group space structure and the associated torsor deformation, one can expect some application of the theory of dual fibrations to the geometry of the original Lagrangian fibration. However substantial applications seem to be out of reach for the time being because of our lack of knowledge, for example, on the geometric or moduli theoretic structure of the (compactified) dual fibration. Also one should note that our method may not apply to Lagrangian fibrations with multiple fibers.

In our Main Theorem, the most important point is the existence of the symplectic form on the dual. In fact the proof of the Main Theorem is almost devoted to the construction of the symplectic form. Our approach toward Main Theorem is based on the explicit description of Lagrangian fibrations due to Matsushita [14] and we construct a symplectic form on the dual fibration *geometrically*. In our proof, the key notion is the *Néron mapping property*, which enables us to apply arguments parallel to those of the geometry of elliptic surfaces. Anyway, our approach much differs from Mukai's elegant cohomological method in [15]. The author does not know whether one can prove our Main Theorem via cohomological or moduli theoretic arguments, although there should be a more or less conceptual proof.

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1. Basic definitions

In this section, we present a review of basic definitions, notations, and facts which are necessary in this paper. In particular we give a definition of the dual fibration of an abelian fibration.

Symplectic manifolds and Lagrangian fibrations.

Definition 1.1. Let X be a smooth complex manifold of dimension $2n$.

(i) A *symplectic form* on X is a d-closed holomorphic 2-form σ_X on X such that σ_X^n is nowhere vanishing. We call (X, σ_X) a (holomorphic) *symplectic manifold*.

(ii) Let (X, σ_X) be a symplectic manifold. A morphism $f : X \to S$ is called a *Lagrangian fibration* if f is of relative dimension n, and $\sigma_{X|F} = 0$ for any irreducible component F of any fiber of f.

(iii) We call a compact symplectic Kähler manifold X *irreducible* if the space of global holomorphic 2-forms on X is spanned by the symplectic form σ_X and X is simply connected.

Remark 1.2. Matsushita [11, 12] gave basic results on projective Lagrangian fibrations. According to his articles, we have the following facts.

1. A projective morphism $f : X \to S$ is called *fiber space structure* on X if $f_*\mathcal{O}_X = \mathcal{O}_S$. If X is a projective irreducible symplectic manifold, a non-trivial, non-birational fiber space structure $f : X \to S$ with projective base S is a Lagrangian fibration. Moreover S is a $\mathbb{Q}$-Fano manifold, *i.e.*, S has only klt singularities and $-K_S$ is ample, and $\rho(S) = 1$.
2. If a Lagrangian fibration f is proper, a general fiber of f is an abelian variety of dimension n.
3. When $f : X \to S$ is a proper morphism with X symplectic, f is a Lagrangian fibration if $\sigma_{X|F} = 0$ for *general* fiber F of f. In particular, f is equidimensional under the same condition.

Picard space. First we recall the definition of Picard functor after [7, 8]. Let $f : X \to S$ be a morphism of analytic spaces. We define the Picard group relative to f as

$$\mathrm{Pic}(X/S) = H^0(S, R^1 f_* \mathcal{O}_X^*).$$

The *Picard functor* $\mathcal{P}ic_{X/S}$: (Analytic spaces$/S$) $\to$ (Sets) is defined by

$$\mathcal{P}ic_{X/S}(T) = \mathrm{Pic}(X_T/T),$$

where $T \to S$ is an analytic space over S, $f_T : X_T \to T$ is the base change of f by $T \to S$ and $\mathrm{Pic}(X_T/T)$ is the Picard group relative to f_T.

For the moduli problem defined by this Picard functor, we consider, as usual, the representability of the functor. For a projective morphism $f : X \to S$, we consider the following conditions.

Condition (P)**.** (i) f is flat. (ii) $f_*\mathcal{O}_X = \mathcal{O}_S$. (iii) f has analytic local sections.

Here we say f *admits analytic local sections* if for any point $s \in S$ there exists an open neighborhood U of s in Euclidean topology such that we have a section $o_U : U \to f^{-1}(U)$ of $f_{|U}$. This is equivalent to assume that any scheme theoretic fiber X_s ($s \in S$) has a reduced component if the total space X is smooth and f is equidimensional.

Under the condition (P), it is a basic fact that there exists the fine moduli (algebraic or analytic) space $\mathrm{Pic}_{X/S} \to S$ which represents the Picard functor $\mathcal{P}ic_{X/S}$, [1, 7]. We call this space the *Picard space* of X/S. By definition, $\mathcal{O}_X$ gives a canonical section $o : S \to \mathrm{Pic}_{X/S}$ and $\mathrm{Pic}_{X/S} \to S$ is a commutative group space (more precisely, a commutative group object in the category of algebraic or analytic spaces) with the identity element o.

Dual fibration of an abelian fibration. For an abelian variety A, $\widehat{A} = \mathrm{Pic}^0_{A/\mathbb{C}}$ is called the *dual abelian variety* of A. We give a relativization of the notion of "dual" for abelian fibrations.

An *abelian fibration* $f : X \to S$ is a projective morphism whose general fiber is an abelian variety. For such a fibration, the identity component $\mathrm{Pic}^0_{X/S} \to S$ of $\mathrm{Pic}_{X/S}$ would be a candidate of the dual, if it exists, of $X \to S$, but this does not suit for our purpose. We define the dual fibration of an abelian fibration following the idea of Raynaud [18], see also [3].

Let $f : X \to S$ be a flat projective morphism, H an f-ample line bundle and F the general fiber of f. Let $T \to S$ be a morphism and $f_T : X_T = X \times_S T \to T$ the base change of f. We define a functor $\mathcal{P}$, which is a sub-functor of $\mathcal{P}ic_{X/S}$ by

$$\mathcal{P}(T) = \{L \in \mathcal{P}ic_{X/S}(T) \mid \chi(n; X_t, L_t) = \chi(n; F, \mathcal{O}_{|F}) \text{ for any } t \in T\}$$

where X_t is the fiber of f_T over t, L_t the image of L by

$$\mathcal{P}ic_{X/S}(T) \to \mathcal{P}ic_{X/S}(\{t\})$$

and

$$\chi(n; X_t, L_t) = \sum_k (-1)^k h^k(X_t, L_t \otimes H^n_{|X_t})$$

the Hilbert polynomial. Note that in this case the condition $\chi(n; L_t) = \chi(n; \mathcal{O}_{|F})$ does not depend on the choice of f-ample line bundle H.

Denote by P, if exists, the space which represents the functor $\mathcal{P}$, which is an analytic subspace of $\mathrm{Pic}_{X/S}$. Note that P in general is not Hausdorff, if there exists a fiber of $f : X \to S$ which is not integral. It is almost equivalent to say that there exists some non-trivial line bundle whose restriction to general fiber is the structural sheaf of the fiber, but not the case on singular fibers. To make the space Hausdorff, we take a quotient by the sub-group space which consists of such sheaves.

Assume that $\mathcal{P}$ is representable, for example, f satisfies the Condition (P). We consider the closure E of the canonical section on the smooth fibers induced by $\mathcal{O}_X$. Then this E is a sub-group space of P. We put $Q = P/E$. Note that natural morphism $E \to S$ is a local isomorphism and the total space Q is Hausdorff.

Definition 1.3. We call the group space $\pi : Q \to S$ the *biggest separated quotient* of P. If $f : X \to S$ is an abelian fibration such that $\mathcal{P}$ is representable, we call $Q \to S$ as above the *dual fibration* of $f : X \to S$.

One of the most important features of the biggest separated quotient of the Picard space is the extension property which one often calls *Néron mapping property*.

Theorem 1.4. *Let $f : X \to S$ be a projective morphism satisfying the condition (P) with X and S smooth, $\Delta \subset S$ a proper closed subset such that f is smooth over $S_0 = S \backslash \Delta$ and $g : Y \to S$ be a smooth morphism. Let $\pi : Q \to S$ be the biggest separated quotient of the Picard space of f as in Definition 1.3. Put $Q_0 = \pi^{-1}(S_0)$ and $Y_0 = g^{-1}(S_0)$. Then any (S_0)-morphism $\alpha_0 : Y_0 \to Q_0$ extends uniquely to an S-morphism $\alpha : Y \to Q$.*

Proof. We include a proof of this theorem here for the convenience of readers, although it is well known to specialists, see [18], Théorème 8.1.4, for example. Notice that the extension property in question is local (in Euclidian topology) so that we can assume that f has a section $s : S \to X$ by the condition (P). Take $X_Y = X \times_S Y$ and let $f_Y : X_Y \to Y$ the natural morphism. Then we have a section $s_Y = (s \circ g, \mathrm{id}) : Y \to X_Y = X \times_S Y$ of f_Y. By virtue of this section, we have the exact sequence

$$0 \longrightarrow \mathrm{Pic}(Y_0) \longrightarrow \mathrm{Pic}((X_Y)_0) \longrightarrow \mathcal{P}ic_{X/S}(Y_0) \longrightarrow 0,$$

where $(X_Y)_0 = f_Y^{-1}(Y_0)$ ([8], V. Corollaire 2.4. See also [3], §8.1, Proposition 4). Thus any section $\alpha_0 : Y_0 \to Q_0 = P_{|S_0}$ is represented by a line bundle $L \in \mathrm{Pic}((X_Y)_0)$. As X_Y is non-singular by our assumption, L extends to a line bundle $\widetilde{L}$ on X_Y. The Hilbert polynomial of $\widetilde{L}$ along fibers of f_Y is constant by the flatness so that $\widetilde{L}$ determines an element $\tilde{\alpha} \in \mathcal{P}(Y)$ and consequently a morphism $\alpha : Y \to Q$. This α is an extension of α_0 by functoriality and its uniqueness is a consequence of the separatedness of Q. Q.E.D.

Toroidal degenerations. For the study of abelian fibrations, toroidal degenerations are some of the basic tools. We give a brief review of the definition of a toroidal degeneration and its elementary properties for the sake of reference.

Definition 1.5 (See [16] for details)**.** Let S be a smooth analytic space and $\Delta = \sum \Delta_i$ a reduced simple normal crossing divisor. Put $U = S - \Delta$. S has a natural stratification $S = \coprod S_I$ with $S_I = \bigcap_{i \in I} \Delta_i - \bigcup_{j \notin I} \Delta_j$. Note that $S_\emptyset = U$ by convention. A morphism of finite type $\pi : \mathcal{A} \to S$ is called a *toroidal degeneration of abelian varieties* if it satisfies the following conditions.

(1) $\mathcal{A} \times_S U \to U$ is a smooth abelian fibration of relative dimension g.
(2) $(\mathcal{A}, \mathcal{A} \times_S U) \to (S, U)$ is a toroidal morphism, i.e., $(\mathcal{A}, \mathcal{A} \times_S U)$ and (S, U) are analytically locally isomorphic to toric varieties ($A \times_S U$ and U are identified with the embedded algebraic tori) and π is analytically locally toric morphism under these isomorphisms.
(3) For each stratum S_I, $\mathcal{A}_I = \mathcal{A} \times_S S_I \to S_I$ is a topologically locally trivial deformation.
(4) $\mathcal{A}_I$ has a stratification $\mathcal{A}_I = \coprod_{\alpha \in \Lambda_I} \mathcal{A}_I^\alpha$ satisfying the following conditions.
    (a) There exists a smooth family of abelian varieties $\varphi_I : \mathcal{B}_I \to S_I$ such that $\pi_{|\mathcal{A}_I^\alpha}$ factors through φ_I^α ; $\mathcal{A}_I^\alpha \to \mathcal{B}_I \to S_I$ for any α.
    (b) For any point $t \in S_I$, the fiber $\mathcal{A}_{I,t}^\alpha$ over t is an extension of $\mathcal{B}_{I,t}^\alpha$ by an algebraic torus :

$$1 \longrightarrow (\mathbb{C}^*)^{g - \dim \mathcal{B}_{I,t}^\alpha} \longrightarrow \mathcal{A}_{I,t}^\alpha \longrightarrow \mathcal{B}_{I,t}^\alpha \longrightarrow 0.$$

Namikawa showed in his paper [16] the existence of toroidal degenerations which compactify a given smooth abelian fibration $\mathcal{A}_U \to U$ under some modest assumptions with any base S with toroidal embedding (S, U) without self-intersection on its boundary. Note also that the stratifications of $\mathcal{A}_I$'s above are given by means of fans corresponding to torus embeddings in Namikawa's construction.

Assume that $\pi : \mathcal{A} \to S$ has a section $o : S \to \mathcal{A}$. Put

$$\mathcal{A}^0 = \mathcal{A} - \bigcup_{o(S) \cap \mathcal{A}_I^\alpha = \emptyset} \mathcal{A}_I^\alpha.$$

Then this $\mathcal{A}^0 \to S$ is a commutative group space with connected fiber whose identity element is o, by its construction, which we call the *identity component* of $\mathcal{A}$ by a slight abuse of terminology. If t is in a stratum S_I of codimension 1 in S, *the dimension l of toric part* of $\mathcal{A}_t^0$ is defined by $l = k - 1$ where k is the degree of unipotency of the monodromy along S_I.

2. Degenerations of Kodaira type

In this section, we review the results of Matsushita [13, 14] on the local structure of a projective Lagrangian fibrations and define a *degeneration of Kodaira type* as an abelian fibration with the structure enjoyed by Lagrangian fibrations.

First we recall the following theorem.

Theorem 2.1 (Matsushita [14], see also [13])**.** *Let $f : X \to S$ be a projective Lagrangian fibration such that S is smooth and the discriminant locus $\Delta \subset S$ of f is a smooth divisor. Take a general point $s \in \Delta$, replace S with a sufficiently small neighborhood of s, and consider a cyclic covering $\mu : T \to S$ such that T is smooth and μ branches along Δ. Let G be the covering transformation group of μ and $\widetilde{\Delta} = \mu^{-1}\Delta$ with reduced scheme structure. Then for a suitable choice of μ, there exists a toroidal degeneration $\tilde{f} : \widetilde{X} \to T$ and a subgroup H of G such that the dimension of toric part of fibers of $\widetilde{X}^0$ over $\widetilde{\Delta}$ is at most 1, $\widetilde{X}$ is smooth, $K_{\widetilde{X}}$ is $\tilde{f}$-trivial, i.e., $K_{\widetilde{X}}$ is a pull-back of a line bundle on T, G acts on $\widetilde{X}$, $\tilde{f}$ is G-equivariant, and we have the commutative diagram*

$$\begin{tikzcd}
 & & & W \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & & \\
X \arrow[r, equal] \arrow[d, "f"'] & X'/G' \arrow[d] & X' \arrow[l, "\rho_1"'] \arrow[d, "f'"'] & & \widetilde{X}/H \arrow[d] & \widetilde{X} \arrow[l, "\nu_1"'] \arrow[d, "\tilde{f}"] \\
S \arrow[r, equal] & T/G & T/H \arrow[l, "\rho_2"'] \arrow[rr, equal] & & T/H & T \arrow[l, "\nu_2"']
\end{tikzcd}$$

where

- *$G' = G/H$.*
- *ν_1, ν_2, ρ_1 and ρ_2 are natural quotient maps such that $\rho_2 \circ \nu_2 = \mu$. ν_1 is étale in codimension one and ρ_1 is étale.*
- *α is a resolution, β is bimeromorphic morphism (possibly identity map), and X' is non-singular.*

Note that this theorem asserts, roughly speaking, that Lagrangian fibrations have a structure which is highly analogous to that of elliptic fibrations, outside of a codimension 2 subset of the discriminant locus, as we will see below.

First we remark that if G' is not trivial in the theorem, $f : X \to S$ must have multiple fiber along Δ, since ρ_1 is étale. As we assume that $f : X \to S$ satisfies the condition (P), we assume $G = H$ in the sequel. This is equivalent to say that ρ_1 and ρ_2 are identity maps.

2.2. We need explicit descriptions of ν_1 and ν_2 where G $(= H)$ is non-trivial. We call this case *additive reduction* by an analogy with elliptic fibrations. The descriptions below are due to Matsushita and reader can find their verifications in his papers [13, 14].

Put $\widetilde{X}_{\widetilde{\Delta}} = \tilde{f}^{-1}(\widetilde{\Delta})$. Let us first consider the case where $\tilde{f} : \widetilde{X} \to T$ is a smooth abelian fibration. In this case G acts on the fiber of $\tilde{f}_{\widetilde{\Delta}} : \widetilde{X}_{\widetilde{\Delta}} \to \widetilde{\Delta}$ by the diagonal matrix $\mathrm{diag}(\zeta, 1, \ldots, 1)$ where ζ is the k-th root of unity (k is the order of G) with respect to a suitable coordinate. From this fact, $\tilde{f}_{\widetilde{\Delta}} : \widetilde{X}_{\widetilde{\Delta}} \to \widetilde{\Delta}$ decomposes as

$$\widetilde{X}_{\widetilde{\Delta}} \xrightarrow{\varepsilon} M \xrightarrow{\gamma} \widetilde{\Delta}$$

where ε is a G-equivariant smooth elliptic fibration and γ is a G-equivariant smooth abelian fibration. H is in fact the maximal subgroup of G which acts trivially on the fibers of γ. Since we have assumed $G = H$, G leave γ invariant and we have

$$\widetilde{X}_{\widetilde{\Delta}}/G \xrightarrow{\bar{\varepsilon}} M \longrightarrow \Delta.$$

Let Σ be the set of points where the action of G has a non-trivial stabilizer on $\widetilde{X}$. Then every connected component of Σ is a multi-section of ε and its image in $\widetilde{X}/G$ is the singular locus, which is a locally trivial deformation of surface quotient singularities locally parameterized by M. Therefore we have the family of minimal resolutions $\alpha : W \to \widetilde{X}/G$. If the singularities are of Du Val type, then we have $X' = W$. If the singularities are strictly klt, then we have a sequence of blowing-downs of families of (-1)-curves $\beta : W \to X' = X$ as shown in the paper of Kodaira [9].

As a result, we have a fairly explicit description of the singular fibers of $f : X \to S$. For $t \in \Delta$, we have a morphism $q : X_t \to M_t$ where M_t is an abelian variety and every fiber of q is isomorphic to one of the Kodaira singular fibers of minimal elliptic surfaces of type I_0^*, II, II^*, III, III^*, IV, or IV^*.

Next we consider the case where $\tilde{f} : \widetilde{X} \to T$ is a toroidal degeneration with 1-dimensional toric part. In this case, an additive reduction actually occurs in our situation when $G = \mathbb{Z}/2\mathbb{Z}$ and it acts as a reflection on the dual graph of the fiber over a point on the discriminant locus (it can be seen from the argument of [13] that if the group G be bigger, we should have multiple fibers over Δ). More precisely, we have a decomposition

$$\widetilde{X}_{\widetilde{\Delta}} \xrightarrow{\varepsilon} M \xrightarrow{\gamma} \widetilde{\Delta},$$

where γ is a G-invariant smooth abelian fibration and ε is a G-equivariant locally trivial family of a cycle of $\mathbb{P}^1$'s. In this case $\widetilde{X}/G$ has a locally trivial family of several A_1-singularities so that we have the crepant resolution $\alpha : X' = W \to \widetilde{X}/G$.

Taking these descriptions into account, let us define the concept of a "degeneration of Kodaira type" for our convenience.

Definition 2.3. An abelian fibration $f : X \to S$ is said to be a *degeneration of Kodaira type* if we have a diagram

(1)
$$\begin{tikzcd} & W \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & & \\ X \arrow[d, "f"'] & & \widetilde{X}/G \arrow[d] & \widetilde{X} \arrow[l, "\nu_1"'] \arrow[d, "\tilde{f}"] \\ S \arrow[rr, equal] & & T/G & T \arrow[l, "\nu_2"'] \end{tikzcd}$$

satisfying the following conditions:

- The total space X is smooth, the base space S is smooth and the discriminant locus Δ is a smooth divisor.
- f admits a section.
- T is smooth and ν_2 is a finite cyclic covering ramifying along $\widetilde{\Delta} = \nu_2^{-1}(\Delta)$. G is the Galois group of ν_2.
- $\tilde{f} : \widetilde{X} \to T$ is either a smooth abelian fibration or a toroidal degeneration with one dimensional toric part such that $K_{\widetilde{X}}$ is $\tilde{f}$-trivial.
- For $\widetilde{X}_{\widetilde{\Delta}} = \tilde{f}^{-1}(\widetilde{\Delta})$, we have a decomposition

(2)
$$\tilde{f}_{|\widetilde{X}_{\widetilde{\Delta}}} : \widetilde{X}_{\widetilde{\Delta}} \xrightarrow{\varepsilon} M \xrightarrow{\gamma} \widetilde{\Delta}$$

  where γ is a G-invariant smooth abelian fibration and ε is a G-equivariant smooth elliptic fibration or a G-equivariant locally trivial family of a cycle of $\mathbb{P}^1$'s.
- $\mathrm{Sing}(\widetilde{X}/G) \subset \widetilde{X}/G$ is a locally trivial family of surface quotient singularities locally parameterized by M and α is the (family of) minimal resolution and β is a bimeromorphic morphism so that K_X is f-trivial.

If G is non-trivial, i.e., f is not the same thing as $\tilde{f}$, f is said to have an *additive reduction*.

An abelian fibration $f : X \to S$ is called *locally of Kodaira type* if for any point $s \in S$, there exists an Euclidian neighborhood V_s of s such that the restriction $f_{|V_s} : X_{V_s} \to V_s$ is of Kodaira type.

Remark 2.4. 1. Let $f : X \to S$ be a projective Lagrangian fibration with analytic local sections and Δ its discriminant locus. Note that Theorem 2.1 asserts that there exists a closed subset Δ_1 of codimension not less than 2 such that $\mathrm{Sing}\,\Delta \subset \Delta_1 \subset \Delta$ and f is locally of Kodaira type over $S_1 = S \setminus \Delta_1$.
2. In the definition, the minimal resolution α is not crepant if the singularities on X/G is not Du Val. Only after some contraction β, K_X becomes f-trivial.

Now we prove the following theorem, although one can easily expect that this theorem holds true by analogy with the result of Néron-Kodaira [17, 9].

Theorem 2.5. *Let $h : Z \to S$ be a projective degeneration of abelian varieties of Kodaira type. Let $Z^\sharp$ be the set of points on Z where h is smooth. Then $h^\sharp : Z^\sharp \to S$, which is the restriction of h to $Z^\sharp$, has a unique group space structure.*

The key of the proof of this theorem is the following proposition:

Proposition 2.6. *Let $f : X \to S$ and $g : Y \to S$ be degenerations locally of Kodaira type and $F : X \dashrightarrow Y$ be an S-bimeromorphic map. Then F is an isomorphism.*

Proof. Assume the contrary, i.e., assume that F is not an isomorphism. Then the indeterminacy of F is at least of codimension 2 because X and Y are smooth. Let H be a relatively ample divisor on Y over S and put $H' = F_*^{-1}H$. Then we have a $(K_X + \varepsilon H')$-extremal contraction by the Cone Theorem (*cf.* [10]), which is in fact a small contraction. Let C be a rational curve which is contracted by this contraction. As we assumed that F is locally of Kodaira type, the deformation of the rational curve C sweeps a divisor which dominates a component of the discriminant locus, which is a contradiction. Therefore F must be an isomorphism. Q.E.D.

Remark 2.7. Proposition 2.6 means that an abelian fibration locally of Kodaira type is the unique relative minimal model over the base just as in the case of minimal elliptic surface.

Proof of Theorem 2.5. This proof is also analogous to the case of minimal elliptic surface. Consider the meromorphic map $\mu : Z \times_S Z \dashrightarrow Z$ defined by $(z, w) \mapsto z - w$ on the generic fiber. It is enough to show that μ is defined everywhere on $Z^\sharp$. Take a point $\xi \in Z^\sharp$ such that $h^\sharp(\xi) \in \Delta$ and its neighborhood $U_\xi \overset{\iota_\xi}{\hookrightarrow} Z^\sharp$. Now consider the bimeromorphic map

$$\tau : Z \times_S U_\xi \dashrightarrow Z \times_S U_\xi$$

defined by $(z, w) \mapsto (z - w, w)$. Noting $\mu_{|Z \times_S U_\xi} = \mathrm{pr}_1 \circ \tau$, it is enough to show that τ is defined everywhere. As we assumed that h is of Kodaira type, $Z \times_S U_\xi \to U_\xi$ is also of Kodaira type. Therefore just applying Proposition 2.6 to τ, we know that τ is in fact an isomorphism. Q.E.D.

Remark 2.8. One should note that the proof of the theorem above utilizes the knowledge of the explicit description of singular fibers of a degeneration of "Kodaira type". In fact, one can determine the group structure of the special fiber of a degeneration of abelian varieties of Kodaira type $f^\sharp : X^\sharp \to S$. Let us illustrate one case. Assume that the fiber of ε in (2) is the elliptic curve E with j-invariant $j = 0$ and $G = \mathbb{Z}/3\mathbb{Z}$ acts on both the coordinate around the discriminant Δ and E

by the multiplication of $\zeta_3 = \exp(2\pi i/3)$ (this corresponds to the case of Kodaira singular fiber of type IV). Then we can expect three cases as following.

Case 1. The singular locus of $\widetilde{X_{\widetilde{\Delta}}}/G$ consists of 3 components. In this case, every component of singular locus is a section of $\widetilde{X_{\widetilde{\Delta}}}/G \to M$. Therefore we can see that $\widetilde{X_{\widetilde{\Delta}}} = E \times M$ and the action of G on E is just a complex multiplication on the fibers. Therefore, $X_\Delta =$ (Kodaira singular fiber of type IV) $\times M$ and $X_t^\sharp \cong (\mathbb{C} \times \mathbb{Z}/3\mathbb{Z}) \times M_t$ where $t \in \Delta$.

Case 2. The singular locus of $\widetilde{X_{\widetilde{\Delta}}}/G$ consists of 2 components, one of which, say B_1, is a section of $\widetilde{X_{\widetilde{\Delta}}}/G \to M$ and the other, B_2, is a bisection. As we assumed $X \to S$ has no multiple fiber, G acts on M trivially. This implies that the inverse image of B_1 is a section of $\varepsilon : \widetilde{X_\Delta} \to M$. Hence $\widetilde{X_\Delta} \cong E \times M$. But the inverse image of B_2 is a bisection of ε which is disjoint from B_1. This implies that B_2 is contained in a fiber of the projection $E \times M \to E$, but this is absurd.

Case 3. The singular locus of $\widetilde{X_{\widetilde{\Delta}}}/G$ is irreducible and is a trisection of $\widetilde{X_{\widetilde{\Delta}}}/G \to M$. Its inverse image B in $\widetilde{X_{\widetilde{\Delta}}}$ is also a trisection of ε. Taking the base change by the natural étale morphism $b : B \to M$, $\widetilde{X_{\widetilde{\Delta}}} \times_M B \cong E \times B$. Therefore $X_t^\sharp \cong \mathbb{C} \times B_t$ where $t \in \Delta$ and $B_t = b^{-1}(M_t)$.

In other cases (consult with [13] for the classification), the group structure of special fibers of $X^\sharp \to S$ can be determined by similar (even easier in some cases) arguments.

Corollary 2.9. *Let $h^\sharp : Z^\sharp \to S$ as in the theorem above. Then $Z^\sharp$ has the Néron mapping property, that is, for any smooth $g : Y \to S$ and S_0-morphism $\alpha_0 : Y_0 \to Z_0^\sharp$, there exists a unique extension $\alpha : Y \to Z^\sharp$, where $S_0 = S \setminus \Delta$, $Y_0 = g^{-1}(S_0)$, and $Z_0^\sharp = h^{\sharp -1}(S_0)$.*

Proof. Define a bimeromorphic map

$$\beta : Z \times_S Y \dashrightarrow Z \times_S Y$$

by $(z, y) \mapsto (z + \alpha_0(y), y)$ $(y \in Y_0)$. By Proposition 2.6, β extends to an automorphism on $Z \times_S Y$. Now define $\alpha : Y \to Z$ as a composition of morphisms

$$Y \xrightarrow{(o\circ g, \mathrm{id})} Z \times_S Y \xrightarrow{\beta} Z \times_S Y \xrightarrow{pr_1} Z,$$

where o is a section of $h : Z \to S$. This α is clearly an extension of α_0. Since $g = h \circ \alpha$ is smooth, where h is the structural morphism $h : Z \to S$, the image of α must be contained in $Z^\sharp$. The uniqueness of the extension α is a consequence of the separatedness of $Z^\sharp$. Q.E.D.

3. Toroidal Degenerations with one dimensional toric part

To give an explicit description of the dual fibration, we need the details of the construction of toroidal degenerations, which was done under considerably general setting in [16]. There Namikawa constructed toroidal degenerations using the degeneration data described by some combinatorics of fans. If we consider the case where the degree of unipotency of the monodromy is 2 (i.e., the dimension of the toric part is 1), the natural choice of degeneration data is known classically (see, for example, [2] Chap. I, §4). We review here the construction of toroidal degenerations with one dimensional toric part in a down-to-earth manner, which is much parallel to the exposition of [2], which treats the case of elliptic curves. We will use the notations in this section freely in the proof of Proposition 4.4.

Let $f : X \to S$ a projective abelian fibration with X and S smooth, and K_X f-trivial. Put $k = \dim S$, $n = \dim(\text{fiber of } f)$, so that $\dim X = n + k$. Assume moreover the discriminant locus Δ of f is a smooth divisor and f is a toroidal degeneration with 1-dimensional toric part, namely for any $t \in \Delta$, X_t has 1-dimensional toric part. Since our question is local in nature, we assume $S \cong D^k$ where D is a unit disc in $\mathbb{C}$, and Δ is defined by $t_1 = 0$ with respect to the coordinate $(t_1, t_2, \dots, t_k)$ of D^k. Moreover we assume that f admits a section. Put $S_0 = S \setminus \Delta$ and its universal covering $\widetilde{S}_0 \cong \mathbb{H} \times D^{n-1} \to S_0$ where $\mathbb{H}$ is the upper half plane. Consider the pull back $\tilde{f}_0 : \widetilde{X}_0 \to \widetilde{S}_0$ of $f_0 : X_0 = f^{-1}(S_0) \to S_0$. Let $\widetilde{H}_0$ be the total space of $(\tilde{f}_0)_* \mathcal{H}om_{\mathcal{O}_{\widetilde{X}_0}}(\Omega^1_{\widetilde{X}_0/\widetilde{S}_0}, \mathcal{O}_{\widetilde{X}_0})$. Since $\tilde{f}_0$ is smooth, $\widetilde{H}_0 \to \widetilde{S}_0$ is a vector bundle of rank $n = \dim(\text{fiber of } f)$. Let $\widetilde{\Lambda}_0 = \bigcup_{\tilde{t} \in \widetilde{S}_0} H_1(\widetilde{X}_{0,\tilde{t}}, \mathbb{Z})$, then we have a natural inclusion

$$J : \widetilde{\Lambda}_0 \to \widetilde{H}_0$$

defined by $J(\alpha_{\tilde{t}}) = \int_{\alpha_{\tilde{t}}}$ and, of course, $\widetilde{X}_0 \cong \widetilde{H}_0 / \widetilde{\Lambda}_0$. We give an explicit description of the construction of $f : X \to S$ using the period map and the monodromy.

Let $\langle \omega_1, \dots \omega_n \rangle$ be a sequence of 1-forms on X such that the restriction of these forms on the fiber X_t ($t \in S_0$) is a basis of $H^0(X_t, \Omega^1_{X_t})$. Let $\tilde{\omega}_i$ be the pull-back of ω_i on $\widetilde{X}_0$ and define $\Phi = \langle \tilde{\omega}_1, \dots, \tilde{\omega}_n \rangle$. We call this Φ a marking of f_0.

Let L be an $\tilde{f}_0$-ample line bundle X and

$$E : \widetilde{\Lambda}_0 \times_{\widetilde{S}_0} \widetilde{\Lambda}_0 \to \mathbb{Z}_{\widetilde{S}_0},$$

the corresponding polarization on $\widetilde{\Lambda}_0$. Take a symplectic basis

$$\Psi = (e_1, \dots, e_n, f_1, \dots, f_n) : \mathbb{Z}^{2n} \times \widetilde{S}_0 \to \widetilde{\Lambda}_0$$

with respect to E. Using this basis, polarization form is written as

$$E = \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix}$$

where $\delta = \mathrm{diag}(d_1, \cdots, d_n)$. The period matrix $\Pi = \begin{pmatrix} \Pi_1(\tilde{t}) \\ \Pi_2(\tilde{t}) \end{pmatrix}$ with respect to Φ and Ψ is defined by

$$(\Pi_1(\tilde{t}))_{ij} = \int_{e_i(\tilde{t})} \tilde{\omega}_j, \quad (\Pi_2(\tilde{t}))_{ij} = \int_{f_i(\tilde{t})} \tilde{\omega}_j.$$

By a suitable choice of the marking Φ, we may assume $\Pi = \begin{pmatrix} \Pi_1(\tilde{t}) \\ I_n \end{pmatrix}$. It is well known that $\tau(\tilde{t}) = \Pi_1(\tilde{t})\delta \in \mathfrak{S}_n$ where $\mathfrak{S}_n$ is the Siegel upper half plane. Again changing Φ, we always use the normal form $\Pi = \begin{pmatrix} \tau(\tilde{t}) \\ \delta \end{pmatrix}$. Let

$$M : \mathrm{Gal}(\widetilde{S}_0/S_0) \to Sp(E, \mathbb{Z}) = \{A \in GL(2n, \mathbb{Z}) \mid AE\,{}^t\!A = E\}$$

the monodromy representation. We write

$$M_g = M(g) = \begin{pmatrix} A_g & B_g \\ C_g & D_g \end{pmatrix}.$$

Define Γ as an extension

$$0 \to \mathbb{Z}^{2n} \to \Gamma \to \mathrm{Gal}(\widetilde{S}_0/S_0) \to 0$$

determined by the action of $\mathrm{Gal}(\widetilde{S}_0/S_0)$ on $\mathbb{Z}^{2n}$, $m = (m_1, \ldots, m_{2n}) \overset{g}{\mapsto} m\,M(g)$. Note that we have a matrix representation of Γ:

$$\Gamma \cong \left\{ \begin{pmatrix} 1 & m_{(1)} & m_{(2)} \\ 0 & A_g & B_g \\ 0 & C_g & D_g \end{pmatrix} \middle|\, m = (m_{(1)}, m_{(2)}) \in \mathbb{Z}^{2n},\ \begin{pmatrix} A_g & B_g \\ C_g & D_g \end{pmatrix} = M(g),\ g \in \mathrm{Gal}(\widetilde{S}_0/S_0) \right\}.$$

From now on, we fix a generator of $\mathrm{Gal}(\widetilde{S}_0/S_0)$ and it is denoted by g unless otherwise stated. The group Γ acts on $\widetilde{H}_0$ by

$$\begin{aligned} m \in \mathbb{Z}^{2n} \qquad &: (v, \tilde{t}) \mapsto (v + m \begin{pmatrix} \tau(\tilde{t}) \\ \delta \end{pmatrix}, \tilde{t}) \\ g \in \mathrm{Gal}(\widetilde{S}_0/S_0) \quad &: (v; \xi, t_2, \ldots, t_k) \mapsto (v(C_g\tau(\tilde{t}) + D_g\delta)^{-1}\delta; \xi + 1, t_2, \ldots, t_k) \end{aligned}$$

where $\tilde{t} = (\xi, t_2, \ldots, t_k) \in \mathbb{H} \times D^{k-1}$. Note that we have the compatibility condition

$$g \cdot \tau(\tilde{t}) = (A_g\tau(\tilde{t}) + B_g\delta)(C_g\tau(\tilde{t}) + D_g\delta)^{-1}\delta = \tau(g \cdot \tilde{t})$$

where $g \cdot \tilde{t} = (\xi + 1, t_2, \ldots, t_k)$. By this action, we have $X_0 \cong \widetilde{H}_0/\Gamma$.

Since the monodromy is unipotent in our case (namely, by the condition that f is toroidal), we may assume after changing the symplectic basis Ψ suitably

$$\tau(\tilde{t}) = \tau'(\tilde{t}) + \frac{\xi}{2\pi i}B_g, \quad B_g = \begin{pmatrix} O_{n-1} & 0 \\ 0 & b \end{pmatrix}$$

and

$$M(g) = \begin{pmatrix} I_n & B_g\delta^{-1} \\ O & I_n \end{pmatrix}, \tag{3}$$

where τ' is periodic with respect to the action of $\mathrm{Gal}(\widetilde{S}_0/S_0)$. Note that (3) implies that $\frac{b}{d_n} \in \mathbb{Z}$. We call this marking Φ and symplectic basis Ψ *normalized*.

Now let Γ_1 be the subgroup of Γ which is generated by

$$\begin{pmatrix} 1 & 0 & e_n \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{pmatrix} \text{ where } e_n = (0, 0, \ldots, 1) \in \mathbb{Z}^n,$$

and $\begin{pmatrix} 1 & 0 \\ 0 & M_g \end{pmatrix}$. Then we know that $\Gamma_1 \cong \mathbb{Z} \oplus \mathbb{Z}$ and Γ_1 is a normal subgroup of Γ so that $\Gamma/\Gamma_1 \cong \mathbb{Z}^{2n-1}$. We have $\widetilde{H}_0/\Gamma_1 \cong \mathbb{C}^{n-1} \times \mathbb{C}^* \times S_0$ and natural map $\widetilde{H}_0 \cong \mathbb{C}^n \times \mathbb{H} \times D^{k-1} \to \widetilde{H}_0/\Gamma_1 \cong \mathbb{C}^{n-1} \times \mathbb{C}^* \times D_0 \times D^{k-1}$ is given by

$$(v_1, \ldots, v_n; \xi, t_2, \ldots, t_k) \mapsto (v_1, \ldots, v_{n-1}, \exp(2\pi i \frac{v_n}{d_n}); \exp(2\pi i \xi), t_2, \ldots, t_k),$$

Put $Y_0 = \mathbb{C}^{n-1} \times \mathbb{C}^* \times D_0 \times D^{k-1}$. Under the isomorphism $\widetilde{H}_0/\Gamma_1 \cong Y_0$, the action of $\Gamma_2 = \Gamma/\Gamma_1$ on Y_0 is described as follows. Let $m = (m_1, \ldots, m_n; m_{n+1}, \ldots m_{2n-1}) \in \Gamma_2 \cong \mathbb{Z}^{2n-1}$. Put $(a(m)_1, \ldots, a(m)_n) = (m_1, \ldots, m_n)\tau'(\tilde{t}) + m_{n+1}(d_1, 0, \ldots, 0) + \cdots + m_{2n-1}(0, \ldots, 0, d_{n-1}, 0)$. Then m operates on Y_0 by

$$\begin{aligned} (4) \quad & (v_1, \ldots, v_{n-1}, \eta; t_1, t_2, \ldots, t_k) \\ & \qquad \mapsto (v_1 + a(m)_1, \ldots, v_{n-1} + a(m)_{n-1}, \exp(2\pi i \frac{a(m)_n}{d_n}) t_1^{m_n \frac{b}{d_n}} \eta; t), \end{aligned}$$

and we have $X_0 \cong Y_0/\Gamma_2$.

We reconstruct X as Y/Γ_2 where Y is a partial compactification of Y_0. Let $\mathcal{Z}$ be the toric variety corresponding to the fan on $\mathbb{R}^2$

$$\Sigma = \left\{ \sigma_l = \{(x, y) \in \mathbb{R}^2 \mid lx \leqslant y \leqslant (l+1)x\} \right\}_{l \in \mathbb{Z}}.$$

Note that $\mathcal{Z}$ has a morphism $p : \mathcal{Z} \to \mathbb{A}^1$ corresponding to the projection of the fan to the x-axis. Put $Z = p^{-1}(D)$, where $D \hookrightarrow \mathbb{A}^1$ is the unit disc centered at $0 \in \mathbb{A}^1$, and $Y = \mathbb{C}^{n-1} \times Z \times D^{k-1} (\supset Y_0)$. The fiber of induced map $Y \to S = D^k$ over a point in $\Delta = (t_1 = 0)$ is (an infinite chain of $\mathbb{P}^1$'s) $\times \mathbb{C}^{n-1}$. The action (4) of Γ_2 on Y_0 extends to an action on Y. Let $\mathcal{V}_l$ be the affine piece of $\mathcal{Z}$ corresponding to σ_k and $V_l = \mathcal{V}_l \cap Z$. Then the action of $(0, \cdots, 1; 0, \cdots, 0)$ brings $\mathbb{C}^{n-1} \times V_l \times D^{k-1}$ to $\mathbb{C}^{n-1} \times V_{l+\frac{b}{d_g}} \times D^{k-1}$ so that the action of Γ_2 on Y is properly discontinuous and fixed point free since we restrict ourselves over the unit disc. Take the quotient $X' = Y/\Gamma_2$. The logarithmic $(n+k)$-form

$$\omega = dv_1 \wedge \cdots \wedge dv_{n-1} \wedge \frac{d\eta}{\eta} \wedge dt_1 \wedge \cdots \wedge dt_k$$

on Y_0 in the coordinate of (4) is invariant under the action of Γ_2 and naturally extends to a nowhere vanishing holomorphic $(n+k)$-form on Y, so that $K_{X'/S}$ is trivial. Since X' constructed in this way is obviously a degeneration of Kodaira type, we know that X' is isomorphic to X over S applying Proposition 2.6.

4. Dual fibration of a degeneration of Kodaira type

Now we proceed to give an explicit description of the dual of a Lagrangian fibration in codimension 1. More precisely, for a degeneration of Kodaira type $f : X \to S$, we define a *projective* morphism $\hat{f} : \widehat{X} \to S$ and show that its restriction to the smooth locus $\widehat{X}^\sharp \to S$ is isomorphic to the dual fibration $Q \to S$. The result can be summarized as the following theorem:

Theorem 4.1. *Let $f : X \to S$ be an abelian fibration of Kodaira type. Then there exists an abelian fibration of Kodaira type $\hat{f} : \widehat{X} \to S$ whose smooth part $\hat{f}^\sharp : \widehat{X}^\sharp \to S$ is isomorphic to the dual fibration $\pi : Q \to S$ of f as a group space.*

Remark 4.2. By Proposition 2.6, this implies that $\hat{f} : \widehat{X} \to S$ is a unique relatively minimal compactification of $\pi : Q \to S$ (see Remark 2.7).

We prove this theorem taking care of the details of the construction of $\hat{f} : \widehat{X} \to S$. We keep $k = \dim S$ and $n = \dim(\text{fiber of } f)$ throughout this section.

4.3. Smooth case. First we construct the dual fibration for a smooth abelian fibration $f : X \to S$. Here we assume that f admits a section, which always exists if we shrink S appropriately.

Let H be the total space of $f_*\mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{O}_X)$ and $\Lambda = \bigcup_{s\in S} H_1(X_s, \mathbb{Z})$ be the locally constant sheaf whose every stalk is free abelian group of rank $2n$. We have the natural inclusion $\Lambda \to H$ by integration $\gamma \mapsto \int_\gamma$, thus we have

$$X \cong H/\Lambda$$

where we regard H and Λ as group spaces over S. Let L be an f-ample line bundle on X. Then L induces alternating bilinear form

$$E : \Lambda \times_S \Lambda \to \mathbb{Z}_S,$$

called polarization, such that its coefficient extension

$$E_\mathbb{R} : (\Lambda \otimes \mathbb{R}) \times_S (\Lambda \otimes \mathbb{R}) \to \mathbb{R}_S$$

is non-degenerate. Taking a symplectic basis, we can represent E by a matrix

$$E = \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix}$$

where $\delta = \operatorname{diag}(d_1, \cdots, d_n)$. Let

$$\Lambda^* = \{l \in \Lambda \otimes \mathbb{R} \mid E(l, \lambda) \in \mathbb{Z} \text{ for any } \lambda \in \Lambda \text{ with } pr(l) = pr(\lambda)\},$$

where pr is the projection $\Lambda \otimes \mathbb{R} \to S$. Then we define $\widehat{X} = H/\Lambda^*$. Note that the morphism $\hat{f} : \widehat{X} \to S$ is smooth and $\hat{f}$ is in fact a group space. We clearly have $Q \cong \widehat{X}$. By this construction, $\widehat{X}$ is an étale quotient of X. On the other hand, one can see $\widehat{X}$ as an étale covering space of X. Let

$$\widehat{\Lambda} = \{l \in \Lambda \mid E(l, \lambda) \in e(\delta)\mathbb{Z} \text{ for any } \lambda \in \Lambda \text{ with } pr(l) = pr(\lambda)\} \subset \Lambda,$$

where $e(\delta) = l.c.m.(d_1, \dots, d_n)$. Then we have also an isomorphism $\widehat{X} \cong H/\widehat{\Lambda}$ and finite étale morphism $\varphi_L : \widehat{X} \to X$. The composition $\widehat{X} \to X \to \widehat{X}$ is nothing but the multiplication by $e(\delta)$ on the fibers.

Toroidal case. Next we consider the case $f : X \to S$ is a toroidal degeneration with 1-dimensional toric part. The fundamental idea of the construction is completely same as in the smooth case. More precisely, we show the following proposition:

Proposition 4.4. *Let $f : X \to S$ be a projective abelian fibration such that X and S are smooth, K_X is f-trivial, the discriminant locus Δ of f is a smooth divisor on S, and f is a toroidal degeneration with 1-dimensional toric part. Then we can construct a toroidal degeneration with 1-dimensional toric part $\hat{f} : \widehat{X} \to S$ over the discriminant locus Δ such that the $\widehat{X}$ is smooth, $K_{\widehat{X}}$ is $\hat{f}$-trivial, and the smooth locus $\widehat{X}^\sharp$ is isomorphic to the dual fibration $Q \to S$ of f over S. Moreover if we have a section of $f : X \to S$, there is an étale morphism $\varphi_L : Q = \widehat{X}^\sharp \to X^\sharp$ which is the isogeny on the generic fiber corresponding to a choice of f-ample line bundle L.*

Proof. The question is local in S so that we may assume a section $s : S \to X$ of $f : X \to S$ from the beginning. Recall the notations in §3. First we dualize the family of lattices $\widetilde{\Lambda}_0$ as

$$\hat{\hat{\Lambda}}_0 = \{l \in \widetilde{\Lambda}_0 \mid E(l, \lambda) \in e(\delta)\mathbb{Z} \text{ for any } \lambda \in \widetilde{\Lambda}_0 \text{ with } pr(l) = pr(\lambda)\}$$

where $e(\delta) = l.c.m.\,(d_1, \dots, d_n)$. We put a polarization on $\hat{\hat{\Lambda}}_0$ by $\widehat{E} = \frac{1}{e(\delta)} E \otimes \mathbb{R}$. Under this polarization, considering $\hat{\hat{\Lambda}}_0 \otimes \mathbb{R} = \widetilde{\Lambda}_0 \otimes \mathbb{R}$, we have a symplectic basis $\widehat{\Psi}$ of $\hat{\hat{\Lambda}}_0$ as $\hat{e}_i = \hat{d}_i e_i$, $\hat{f}_i = \hat{d}_i f_i$ $(i = 1, 2, \dots, n)$ for $\hat{d}_i = e(\delta) d_i^{-1}$ and the polarization matrix

$$\widehat{E} = \begin{pmatrix} 0 & \hat{\delta} \\ -\hat{\delta} & 0 \end{pmatrix} = e(\delta) E^{-1}, \text{ where } \hat{\delta} = \mathrm{diag}(\hat{d}_1, \dots \hat{d}_n).$$

Using the natural inclusion by integration

$$\widehat{J} = J \otimes \mathbb{R}_{|\hat{\hat{\Lambda}}_0} : \hat{\hat{\Lambda}}_0 \to \widetilde{H}_0,$$

we get the period matrix of dualized family as

$$\widehat{\Pi} = \begin{pmatrix} \hat{\delta}\tau(\tilde{t}) \\ \hat{\delta}\delta \end{pmatrix} = \begin{pmatrix} \hat{\delta}\tau(\tilde{t}) \\ e(\delta) I_n \end{pmatrix}$$

under the normalized Φ and Ψ. The monodromy matrix with respect to $\widehat{\Psi}$ is

$$\widehat{M}(g) = \begin{pmatrix} \hat{\delta} & 0 \\ 0 & \hat{\delta} \end{pmatrix} M(g) \begin{pmatrix} \hat{\delta}^{-1} & 0 \\ 0 & \hat{\delta}^{-1} \end{pmatrix} = \begin{pmatrix} I_n & B_g \delta^{-1} \\ O & I_n \end{pmatrix} (= M(g)), \quad B_g = \begin{pmatrix} O_{n-1} & 0 \\ 0 & b \end{pmatrix}.$$

Therefore the dualized action of $\widehat{\Gamma} \cong \Gamma$ on $\widetilde{H}_0$ is given by

$$\begin{aligned} m \in \mathbb{Z}^{2n} \quad &: (v, \tilde{t}) \mapsto (v + m\begin{pmatrix} \hat{\delta}\tau(\tilde{t}) \\ e(\delta)I_n \end{pmatrix}, \tilde{t}) , \\ g \in \mathrm{Gal}(\widetilde{S}_0/S_0) \quad &: (v; \xi, t_2, \ldots, t_k) \mapsto (v; \xi + 1, t_2, \ldots, t_k) . \end{aligned}$$

Here note that $C_g = O$ and $D_g = I_n$ in the monodromy matrix since Φ and $\widehat{\Psi}$ are normalized. Consider the quotient $\widetilde{H}_0/\widehat{\Gamma}_1 \cong \mathbb{C}^{n-1} \times \mathbb{C}^* \times S_0 \,(\cong Y_0)$ by the subgroup $\widehat{\Gamma}_1 \cong \Gamma_1$. In this case, the natural map $\widetilde{H}_0 \cong \mathbb{C}^n \times \mathbb{H} \times D^{k-1} \to \widetilde{H}_0/\widehat{\Gamma}_1 \cong \mathbb{C}^{n-1} \times \mathbb{C}^* \times D_0 \times D^{k-1}$ is given by

$$(v_1, \ldots, v_n; \xi, t_2, \ldots, t_k) \mapsto (v_1, \ldots, v_{n-1}, \exp(2\pi i \frac{v_n}{e(\delta)}); \exp(2\pi i \xi), t_2, \ldots, t_k).$$

and $\widehat{\Gamma}_2 = \widehat{\Gamma}/\widehat{\Gamma}_1 \cong \mathbb{Z}^{2n-1}$ acts on $\widehat{Y}_0 = \mathbb{C}^{n-1} \times \mathbb{C}^* \times S_0 \,(\cong Y_0)$ by

$$\begin{aligned} (5) \quad & (v_1, \ldots, v_{n-1}, \eta; t_1, t_2, \ldots, t_k) \\ & \qquad \mapsto (v_1 + \hat{a}(m)_1, \ldots, v_{n-1} + \hat{a}(m)_{n-1}, \exp(2\pi i \frac{\hat{a}(m)_n}{e(\delta)}) t_1^{m_n \frac{b}{d_n}} \eta; t). \end{aligned}$$

where

$$\begin{aligned} (\hat{a}(m)_1, \ldots, \hat{a}(m)_n) = (m_1, \ldots, m_n)\hat{\delta}\tau'(\tilde{t}) + e(\delta)m_{n+1}(1, 0, \ldots, 0) + \cdots \\ + e(\delta)m_{2n-1}(0, \ldots, 0, 1, 0) \end{aligned}$$

for $m = (m_1, \ldots, m_n; m_{n+1}, \ldots, m_{2n-1}) \in \widehat{\Gamma}_2$. We take the same partial compactification $\widehat{Y} = \mathbb{C}^{n-1} \times Z \times D^{k-1} \supset \widehat{Y}_0$ as $Y \supset Y_0$ before. Then, the action (5) extends to an action on $\widehat{Y}$. We put $\widehat{X} = \widehat{Y}/\widehat{\Gamma}_2$. Note that this construction is compatible with the construction of $\widehat{X}$ in the smooth case. By this construction, $K_{\widehat{X}/S}$ is also trivial so that $\widehat{X}^\sharp$ is of Kodaira type and has the Néron mapping property (Corollary 2.9). Therefore combining the Néron mapping property of Q (Theorem 1.4), we get $Q \cong \widehat{X}^\sharp$.

On $S_0 = S \setminus \Delta$, we have a finite étale S_0-morphism $\varphi_{L,0} : Q_0 = \pi^{-1}(S_0) \to X_0 = f^{-1}(S_0)$. This $\varphi_{L,0}$ is also a homomorphism by the existence of the section s. It is implied from Theorem 2.5 and Corollary 2.9 that the smooth part $X^\sharp \to S$ is a group space with its identity element s and $\varphi_{L,0}$ extends to a unique S-homomorphism $\varphi_L : Q \to X^\sharp$.

As is well known, the identity component $X^{\sharp,0}$ has a good description as in the smooth case: let $H = T_{X/S|s(S)}$ be the total space of the relative tangent bundle restricted to the section s and $\Lambda = \bigcup_{s \in S} H_1(X_s, \mathbb{Z})$, which is a subsheaf of H, but not locally constant. In fact $\Lambda_0 = \Lambda_{|S_0}$ is locally constant of rank $2n$ and $\Lambda_{|\Delta}$ is locally constant of rank $2n-1$. Here we have $X^{\sharp,0} \cong H/\Lambda$ by the exponential map. On Λ_0, we have a polarization $E : \Lambda_0 \times_{S_0} \Lambda_0 \to \mathbb{Z}_{S_0}$. Put

$$\widehat{\Lambda}_0 = \{l \in \Lambda_0 \mid E(l, \lambda) \in e(\delta)\mathbb{Z} \text{ for any } \lambda \in \Lambda_0 \text{ with } pr(l) = pr(\lambda)\}$$

where $e(\delta)$ is the same as the one in the paragraph 4.3, and let $\widehat{\Lambda}$ be the closure of $\widehat{\Lambda}_0$ in Λ. This $\widehat{\Lambda}$ is naturally a subsheaf of Λ and we have $\widehat{X}^{\sharp,0} \cong Q^0 \cong H/\widehat{\Lambda}$, where Q^0 is the identity component of Q, and an étale morphism $\psi : Q^0 \to X^{\sharp,0} \subset X^{\sharp}$. By the uniqueness of the extension (i.e., the separatedness of $X^{\sharp,0}$), ψ is nothing but φ_L restricted to Q^0. Therefore, we know that φ_L is in fact an étale morphism. Q.E.D.

Additive reduction case. Finally consider the case $f : X \to S$ is a degeneration of Kodaira type with additive reduction. We show the following:

Proposition 4.5. *Let $f : X \to S$ be an abelian fibration of Kodaira type with additive reduction and take the diagram* (1)*. Then we have an action of G on the compactified dual $\hat{\tilde{f}} : \hat{\tilde{X}} \to T$ and the types of the singularities on $\widetilde{X}/G$ and $\hat{\tilde{X}}/G$ are the same so that we get an abelian fibration of Kodaira type $\hat{f} : \widehat{X} \to S$ with the reduction diagram*

$$
\begin{tikzcd}
 & \widehat{W} \arrow[dl, "\hat{\beta}"'] \arrow[dr, "\hat{\alpha}"] & & \\
\widehat{X} \arrow[d, "\hat{f}"'] & & \hat{\tilde{X}}/G \arrow[d] & \hat{\tilde{X}} \arrow[l, "\hat{\nu}_1"'] \arrow[d, "\hat{\tilde{f}}"] \\
S \arrow[rr, equal] & & T/G & T, \arrow[l, "\nu_2"']
\end{tikzcd} \tag{6}
$$

such that its smooth part $\hat{f}^{\sharp} : \widehat{X}^{\sharp} \to S$ is isomorphic to the dual fibration $\pi : Q \to S$ of f. Moreover, $\hat{\alpha}^* K_{\hat{\tilde{X}}/G} - \hat{\beta}^* K_{\widehat{X}} = c\hat{\alpha}^*(\hat{\tilde{X}}_{\Delta}/G)$ for the non-negative integer c determined by the formula $\alpha^* K_{\widetilde{X}/G} - \beta^* K_X = c\alpha^*(\widetilde{X}_{\Delta}/G)$. In particular $K_{\widehat{X}} = 0$ if and only if $K_X = 0$.

Proof. Recall that The fibration $\tilde{f} : \widetilde{X} \to T$ in the diagram (1) is either a smooth abelian fibration or a toroidal degeneration with 1-dimensional toric part. We have a section of f by the definition of a degeneration of Kodaira type. Moreover, shrinking S, we may always assume that $\tilde{f}$ has a section. By the Néron mapping property for $\tilde{f}^{\sharp}$ (Corollary 2.9), the section of f induces a section of $\tilde{f}$ which is invariant under the action of $G = \mathrm{Gal}(T/S)$. We already know that there exists the *compactified* dual fibration $\hat{\tilde{f}} : \hat{\tilde{X}} \to T$ of $\tilde{f}$ (the paragraph 4.1 and Proposition 4.4). G acts on $\hat{\tilde{f}} : \hat{\tilde{X}} \to T$ by the monodromy of $\tilde{f}_0 : \widehat{X}_0 \to S_0$, which is the dual of $f_0 : X_0 \to S_0$.

First, consider the case where $\tilde{f}$ is a smooth abelian fibration. In this case $\hat{\tilde{f}}$ is also smooth. We have an étale morphism $\varphi_L : \hat{\tilde{X}} \to \widetilde{X}$ which is G-equivariant. Therefore we have G-equivariant étale morphism $\hat{\tilde{X}}_{\widetilde{\Delta}} \to \widetilde{X}_{\widetilde{\Delta}}$, which is the restriction of φ_L. In fact, this induces an isogeny on the fibers over $\widetilde{\Delta}$. As G acts on the fiber of $\widetilde{X}_{\widetilde{\Delta}} \to \widetilde{\Delta}$ by $\mathrm{diag}(\zeta, 1, \ldots, 1)$, where ζ is a root of unity, G also acts on the fiber of $\hat{\tilde{X}}_{\widetilde{\Delta}} \to \widetilde{\Delta}$ by this matrix. This, in fact, implies that we have a decomposition $\hat{\hat{X}}_{\widetilde{\Delta}} \xrightarrow{\hat{\varepsilon}} \hat{M} \xrightarrow{\hat{\gamma}} \widetilde{\Delta}$ with G-invariant $\hat{\gamma}$ and G-equivariant $\hat{\varepsilon}$ as in the paragraph 2.2. Therefore the structures of the locus of the points whose stabilizer is non-trivial on $\widetilde{X}$ and $\hat{\hat{X}}$ are completely the same. In other word, the types of singularities of $\widetilde{X}/G$ and $\hat{\hat{X}}/G$ are same. Thus we have the resolution and contraction process $\hat{\alpha}, \hat{\beta}$ on $\hat{\hat{X}}/G$ as in the diagram (6). As a result, we have an abelian fibration of Kodaira type $\hat{f} : \widehat{X} \to S$ whose generic fiber is the dual of the generic fiber of $f : X \to S$.

Secondly, we consider the case $\tilde{f} : \widetilde{X} \to T$ is a toroidal degeneration with 1-dimensional toric part. Then by Proposition 4.4, we have the compactified dual fibration $\hat{\hat{f}} : \hat{\hat{X}} \to T$, which is also a toroidal degeneration with 1-dimensional toric part. Consider the restriction of $\varphi_L : \hat{\hat{X}}^\sharp \to \widetilde{X}^\sharp$ in Proposition 4.4 to $\hat{\hat{X}}^{\sharp,0}_{\widetilde{\Delta}}$. Then, we get the following diagram.

$$\begin{array}{ccc}
\hat{\hat{X}}^{\sharp,0}_{\widetilde{\Delta}} & \xrightarrow{\varphi'} & \widetilde{X}^{\sharp,0}_{\widetilde{\Delta}} \\
\Big\downarrow{\scriptstyle \hat{\varepsilon}^\sharp} & & \Big\downarrow{\scriptstyle \varepsilon^\sharp} \\
\widehat{B} & \xrightarrow{\varphi''} & B \\
\Big\downarrow{\scriptstyle \hat{\gamma}} & & \Big\downarrow{\scriptstyle \gamma} \\
\widetilde{\Delta} & = & \widetilde{\Delta}
\end{array},$$

where $\widetilde{X}^{\sharp,0}$ and $\hat{\hat{X}}^{\sharp,0}$ are the identity components of $\widetilde{X}^\sharp$ and $\hat{\hat{X}}^\sharp$ as before, $\hat{\varepsilon}^\sharp$ and $\varepsilon^\sharp$ are $\mathbb{C}^*$-fibrations, $\hat{\gamma}$ and γ are smooth abelian fibrations, and φ' and φ'' are induced morphisms by φ_L. Here we know that ε is G-equivariant and G acts on B trivially. By the description of $\psi = \varphi_{L|\hat{\hat{X}}^{\sharp,0}}$, we know that φ'' is an isogeny. Therefore G acts on $\widehat{B}$ trivially. Since G fixes the 0-section of $\hat{\hat{X}}^\sharp$, G acts on $\hat{\hat{X}}$ by the reflection the fibers of $\hat{\hat{X}}_{\widetilde{\Delta}} \to \widetilde{B}$, which is a family of a cycle of $\mathbb{P}^1$'s. Hence $\hat{\hat{X}}/G$ has a family of A_1-singularities so that we have the crepant resolution $\hat{\alpha} : \widehat{X} = \widehat{W} \to \hat{\hat{X}}/G$ which fits into the diagram (6). This completes the proof of the first claim in the proposition.

The second part, which is about the discrepancies, is just elementary case by case computation completely parallel to the case of elliptic surfaces. We note that β is identity unless the singular fiber is of type II, III, or IV. In this case, α is crepant and $c = 0$. In the case of type II, III, and IV singular fiber, corresponding c is 4, 2, and 1, respectively. The last claim is just a consequence of étaleness of the morphism $\varphi_L : \hat{\hat{X}}^\sharp \to \widetilde{X}^\sharp$ in the paragraph 4.3 and Proposition 4.4 because the codimensions of $\hat{\hat{X}} \backslash \hat{\hat{X}}^\sharp$ and $\tilde{X} \backslash \tilde{X}^\sharp$ are at least 2. Q.E.D.

Conclusion of the proof of Theorem 4.1. $\hat{f} : \widehat{X} \to S$ constructed above in each case (the paragraph 4.3, Propositions 4.4 and 4.5) is clearly of Kodaira type. On the other hand, $\hat{f}_0 : \widehat{X}_0 = \hat{f}^{-1}(S_0) \to S_0$ is isomorphic to $Q_0 \to S_0$ as we have seen in the smooth case. Therefore, applying Theorem 1.4 and Corollary 2.9, we get $\widehat{X^\sharp} \cong Q$. Q.E.D.

5. Symplectic form on the dual fibration

In this section, we prove our main theorem.

Theorem 5.1. *Let X be a symplectic Kähler manifold and $f : X \to S$ a projective Lagrangian fibration which has analytic local sections and assume that $R^1 f_* \mathcal{O}_X$ is locally free. Let $\pi : Q \to S$ be the dual fibration of f. Then there exists a natural symplectic form on Q. Moreover, the dual fibration $\pi : Q \to S$ is also a (non-proper, in general) Lagrangian fibration with respect to this symplectic form.*

Remark 5.2. 1. In this theorem, S is automatically smooth, since X is smooth and f admits a local section at any point in S.
2. By the Remark 1.2.3, f is automatically flat, for X and S are smooth, and the condition (P) is satisfied under the assumption of the theorem.
3. If X is projective, the condition $R^1 f_* \mathcal{O}_X$ *is locally free* is automatically satisfied thanks to the theorem of Matsushita [14]: $R^1 f_* \mathcal{O}_X = \Omega^1_S$.

In fact, we can even prove the theorem in a slightly generalized form as in the following.

Let $\mathcal{O}(Q)$ be the sheaf of holomorphic sections of Q over S and take a cohomology class $\lambda \in H^1(S, \mathcal{O}(Q))$. It is well known that a class λ corresponds to a (Q/S)-torsor, i.e., an isomorphism class of $\pi_\lambda : Q^\lambda \to S$ such that there exists an open covering $S = \bigcup_i U_i$ and isomorphism $Q^\lambda_{|U_i} \cong Q_{|U_i}$ over U_i. We call this Q^λ a *twisted dual* of $X \to S$ for our convenience. Of course Q^0 is nothing but the "non-twisted" dual Q.

Theorem 5.3. *Let X be a symplectic Kähler manifold and $f : X \to S$ be a projective Lagrangian fibration which has analytic local sections and assume that $R^1 f_* \mathcal{O}_X$ is locally free. Let $\pi_\lambda : Q^\lambda \to S$ be a twisted dual of f. Then,*

(i) Let Δ be the discriminant locus of X. Then there exists a closed subset Δ_1 of codimension not less than 2 *such that* $\operatorname{Sing}\Delta \subset \Delta_1 \subset \Delta$ *and we have a unique compactification $\hat{f}^\lambda_1 : \widehat{X}^\lambda_1 \to S_1$ of $\pi_\lambda : Q^\lambda_1 \to S_1$ such that $K_{\widehat{X}^\lambda_1}$ is $\hat{f}^\lambda_1$-trivial, where $S_1 = S \setminus \Delta_1$ and $Q^\lambda_1 = \pi_\lambda^{-1}(S_1)$. Moreover $\hat{f}^\lambda_1$ is a degeneration locally of Kodaira type.*

(ii) Let $\widehat{X}^\lambda_2$ be the union of Q^λ and $\widehat{X}^\lambda_1$, which is a non-singular partial compactification of Q^λ. Then there exists a natural symplectic form on $\widehat{X}^\lambda_2$ such that the natural fibration $\hat{f}^\lambda_2 : \widehat{X}^\lambda_2 \to S$ is a Lagrangian fibration with respect to the symplectic form.

We prove this theorem using the explicit description of Q, namely, the construction of $\widehat{X}$. Note that (i) in Theorem 5.3 is just a verbatim sentence of Theorem 4.1 (and its subsequent remark), since a projective Lagrangian fibration is locally of Kodaira type along general smooth points of the divisorial component of the discriminant locus Δ (Theorem 2.1, Remark 2.4). Therefore it is enough to prove (ii). The main point is that Q can be considered as a kind of covering of X locally in S thanks to a polarization on $f : X \to S$.

5.4. Smooth case. First we consider the case $f : X \to S$ is smooth. The proof in this case sounds stupid, but it explains well the fundamental idea of the proof of Theorem5.1. More precisely, we will show the following claim:

Claim. Let X be a symplectic manifold and $f : X \to S$ a smooth projective Lagrangian fibration. Then its twisted dual fibration $\pi_\lambda : Q^\lambda \to S$ is a Lagrangian fibration with respect to a natural symplectic form on Q^λ induced from X.

We should also note that this claim was already known by Donagi and Markman [6].

Proof. First we prove the claim for $Q = Q^0$. Let us assume that f has a section. Then by the paragraph 4.3 in the previous section, we have an étale S-morphism $\varphi_L : Q \to X$ which is defined by means of the polarization on $f : X \to S$. We just take $\sigma_Q = \varphi_L^* \sigma_X$, then this σ_Q must be a symplectic form with respect to which π is Lagrangian.

For general f, we take an open covering $S = \bigcup_{i \in I} U_i$ such that there exists a Lagrangian section $s_i : U_i \to X$ (this is always possible for small enough U_i's). Then $Q_i = \pi^{-1}(U_i)$ admits a natural symplectic form σ_{Q_i} induced from σ_X and $Q_i \to U_i$ is Lagrangian with respect to σ_{Q_i}. Since s_i is a Lagrangian section, σ_X is invariant under the translation by s_i. In fact, let $\varphi_i : Q_i \to X_i$ be the étale morphism corresponding to φ_L, $Q_{ij} = Q_i \cap Q_j$, $X_{ij} = X_i \cap X_j$, and $\psi_{ij} : X_{ij} \to X_{ij}$ the translation by s_j whence s_i is regarded as the 0-section of $X_{ij} \to U_i \cap U_j$. Then we have the following commutative diagram

$$\begin{tikzcd} & Q_{ij} \arrow[dl, "\varphi_i"'] \arrow[dr, "\varphi_j"] & \\ X_{ij} \arrow[rr, "\psi_{ij}"] & & X_{ij} \end{tikzcd}.$$

In this notation, we have

$$\psi_{ij}^* \sigma_X = \sigma_X + f^* s_j^* \sigma_X - f^* s_i^* \sigma_X,$$

here we used the fact that the fibers of $X \to S$ are Lagrangian. As we assumed that s_i are Lagrangian sections, we have $\psi_{ij}^* \sigma_X = \sigma_X$. Therefore $\sigma_{Q,i} = \varphi_i^* \sigma_X =$

$\varphi_j^*\sigma_X = \sigma_{Q,j}$ on $\pi^{-1}(U_i \cap U_j)$. Thus we get σ_Q globally on Q as a patching of $\sigma_{Q,i}$'s.

Also in the twisted case, $\sigma_{Q,i}$ can be patched into a global symplectic form on Q^λ because they are invariant under the translation by Lagrangian sections. Q.E.D.

5.5. Toroidal case. We proceed to prove the theorem in the case of toroidal degeneration with 1-dimensional toric part. Let $f : X \to S$ be a projective Lagrangian fibration which is a toroidal degeneration with 1-dimensional toric part. Here, we assume that f has a section. By Proposition 4.4, we have the fibration $\hat{f} : \widehat{X} \to S$ such that $\hat{f}$ is also toroidal degeneration with 1-dimensional toric part and its smooth locus $\widehat{X}^\sharp \to S$ is nothing but the dual fibration $\pi : Q \to S$. We also know that we have an étale morphism $\varphi_L : Q \to X^\sharp$ which is an extension of the isogeny on the generic fiber induced by the polarization. Therefore $\sigma_Q = \varphi_L^*\sigma_X$ is a symplectic form on Q which is an extension of σ_{Q_0} given in the last paragraph. Therefore we get a symplectic form on Q_1^λ. Note that this symplectic form extends to a symplectic form on $\widehat{X}_1^\lambda$, since the codimension of $\widehat{X}_1^\lambda \backslash Q_1^\lambda$ in $\widehat{X}_1^\lambda$ is greater than 1 and $\widehat{X}_1^\lambda$ is non-singular.

5.6. Additive reduction case. Let us treat the case where $f : X \to S$ is of Kodaira type with additive reduction. Then we have the diagram (1) and (6). Consider $\sigma_{\widetilde{X}} = \nu_1^*\alpha_*\beta^*\sigma$, where σ is the symplectic form on X for which f is Lagrangian. The 2-form $\sigma_{\widetilde{X}}$ is not necessarily symplectic but still symplectic on $\widehat{X}_0$. By the same arguments as in the paragraphs 5.4 and 5.5, $\sigma_{\widetilde{X}}$ determines a 2-form $\sigma_{\hat{\hat{X}}}$ on $\hat{\hat{X}}$. Put $\sigma_{\widehat{X}} = \hat{\beta}_*\hat{\alpha}^*\hat{\nu}_{1*}\sigma_{\hat{\hat{X}}}$. As we see that $K_{\widehat{X}} = 0$ by Proposition 4.5, $\sigma_{\widehat{X}}^{\wedge n}$ is nowhere vanishing. This is equivalent to say that $\sigma_{\widehat{X}}$ is symplectic. As before, $\hat{f}$ is clearly Lagrangian with respect to $\sigma_{\widehat{X}}$. This immediately implies that we have a symplectic form on $\widehat{X}_1^\lambda$ for the general case.

Conclusion of the proof of Theorem 5.3. Take $f : X \to S$ as in Theorem 5.1. As we assumed that $R^1f_*\mathcal{O}_X$ is locally free, the Picard space $\mathrm{Pic}(X/S) \to S$ is smooth morphism so that $\pi : Q \to S$ is smooth. In fact, $R^1f_*\mathcal{O}_X \cong T_{Q/S\,|s(S)}$ and we have an exact sequence

$$0 \longrightarrow T_{Q/S,q} \longrightarrow T_{Q,q} \longrightarrow T_{S,\pi(q)} \qquad (\forall q \in s(S)).$$

so that $\dim T_{Q,q} \le 2n$. On the other hand we have $\dim Q = 2n$, therefore Q is non-singular at q so that π is smooth at q. Consequently, Q^λ is non-singular and $\pi_\lambda : Q^\lambda \to S$ is smooth.

Let $f_0 : X_0 \to S_0$ where $X_0 = f^{-1}(S_0)$, $S_0 = S\backslash\Delta$. By the paragraph 5.4, we have a symplectic form $\sigma_{Q_0^\lambda}$ on $Q_0^\lambda = \pi_\lambda^{-1}(S_0)$ and $\pi_{0,\lambda} : Q_0^\lambda \to S_0$ is Lagrangian with respect to $\sigma_{Q_0^\lambda}$. The paragraphs 5.5 and 5.6 assert that σ_{Q_0} extends to a symplectic form $\sigma_{\widehat{X}_1^\lambda}$ on $\widehat{X}_1^\lambda$ with respect to which $\hat{f}_1^\lambda : X_1^\lambda \to S_1$ is Lagrangian. Since $\widehat{X}_2^\lambda$ is smooth and the codimension of $\widehat{X}_2^\lambda \backslash \widehat{X}_1^\lambda$ in $\widehat{X}_2^\lambda$ is at least 2, $\sigma_{\widehat{X}_1^\lambda}$ extends to a symplectic form on $\widehat{X}_2^\lambda$. Since $\pi_\lambda : Q^\lambda \to S$ is smooth, π_λ is also Lagrangian. Therefore $\hat{f}_2^\lambda : \widehat{X}_2^\lambda \to S$ is a Lagrangian fibration with respect to the symplectic form. This completes the proof of Theorem 5.3, and therefore that of Theorem 5.1. Q.E.D.

Remark 5.7. Cho, Miyaoka and Shephered-Barron asserted the following claim as a theorem: *If $f : X \to S$ is a Lagrangian fibration on a projective irreducible symplectic manifold X over a projective base S admitting a global section, S is the projective space of dimension* $\frac{1}{2}\dim X$ (Theorem 7.2 in [4]). Their argument could apply without any major modifications to our dual fibration, by the existence of a symplectic form in our Theorem 5.1, and we would conclude that the base is the projectve space even if we relax the condition of the existence of a global section to the existence of analytic local sections. In fact, this "application" was one of the motivation of this research. But, unfortunately, the author realized that their proof in [4] does not appear to work at some points after proving our Main Theorem. This means that the theorem of Cho, Miyaoka, Shephered-Barron (or rather the conjecture) seems to be still open even under the original setting of [4].

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