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Dynamic Scaling Analysis of Critical Behaviors in Nonequilibrium Processes

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We present a method for analyzing the critical behaviors systematically in nonequilibrium processes by using dynamic scaling, where we extend the well-known finite-size scaling (FSS) theory for the time evolution of major physical quantities that can indicate either a phase transition or some scaling property. Particularly, we discuss two cases: one is the one-dimensional (1D) thin film growth by vapor deposition polymerization (VDP), and the other is the synchronization of globally-coupled oscillators. Using a dynamic scaling analysis, we show that the universality issue of critical behaviors in nonequilibrium processes can be investigated even though the system is neither in the steady-state limit nor in the thermodynamic limit. Finally, in the context of this extended FSS analysis, we compare the VDP growth with the modified 1D Kardar Parisi-Zhang-type growth and classify the characteristics of synchronization transitions with various setups.

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I. INTRODUCTION

Scaling properties are ubiquitous in nature with real systems far from equilibrium. As tuning the control parameter of the system, it may undergo a nonequilibrium phase transition from one phase to the other one. Moreover, it is well known that the system exhibits collective behaviors near and at the criticality, where the correlation length becomes diverging and covers all over the system. From theoretical point of view, it is very interesting and important how to classify various physical properties of such phase transitions in practice. In order to figure out the thermodynamic limiting behavior of the order parameter in the steady state, one can employ the finite-size-scaling (FSS) theory from equilibrium one.

However, in the absence of analytically exact solutions for physical processes, numerical tests are inevitable, which are unavoidably limited to finite systems and computing facilities. Such an issue has long been already recognized in the context of phase transitions even in equilibrium that the limitation can be exploited [1] to yield insight into the transition nature and the FSS effect. This concept has been already extended to investigate dynamic phase transitions in nonequilibrium systems [2,3]. When a continuous transition contains some crossover behaviors because of nontrivial finite-size correction to scaling, such a careful FSS analysis is particularly valuable since it is governed by the FSS exponent.

Although the FSS analysis is a quite powerful tool to resolve continuous phase transitions and the universality, some technical difficulty remains to obtain enough data in reasonable sizes. It is because numerical simulations take quite long CPU time until the system reaches its steady state. The bigger system the more CPU time algebraically. Due to this, one might analyze incomplete data in small systems that result in some wrong conclusion. To avoid such a misleading analysis, a variety of side techniques are available, such as the higher moment analysis of the order parameter. However, most of them still require the steady-state limit. Thus, it is necessary to find a systematic analysis for temporal behaviors as well as the relaxation of collective behaviors near the saturation at the criticality. This corresponds to the extension of FSS with dynamic scaling [2,3].

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In this paper, we show the dynamic scaling form in the time evolution of major physical quantities for kinetic roughening and synchronization, respectively. Particularly, we discuss surface fluctuations and height-height correlation functions in the one-dimensional (1D) thin film growth of vapor deposition polymerization (VDP) [4,5], and the phase order parameter and its higher order moments in Kuramoto-type coupled oscillators [6,7].

This paper is organized after introduction as follows: In Sec.II, we briefly review the FSS theory and show how to expend it with the time evolution of major physical quantities near and at the criticality, which is numerically checked in Sec.III. Finally, we conclude the paper in Sec. IV with some remarks.

II. EXTENSION OF FINITE-SIZE SCALING AND UNIVERSALITY ISSUE

1. Kinetic Roughening of 1D VDP Growth

Consider a 1D growing surface at the criticality, which can be described by $h(i,t)$ at site i at time t with the initially flat surface: $h(i,0)=0$. To discuss kinetic roughening of the surface, we define surface fluctuations as follows:

$$W^2(L,t) \equiv \frac{1}{L}\sum_{i=1}^{L}[h(i,t)-\bar{h}(t)]^2 = b^{2\alpha}W^2(b^{-1}L, b^{-z}t), \tag{1}$$

where L is the system size, $\bar{h}(t)=\frac{1}{L}\sum_{i=1}^{L}h(i,t)$ is the average height at time t, and b is the length-scale factor. There are two independent critical exponents in surface fluctuations: α is the roughness exponent and z is the dynamic exponent, which determine the universality class of kinetic roughening.

Equation (1) can be rewritten as the following scaling forms, which focus on two different scaling regimes,

$$\begin{aligned} W^2(L,t) &= t^{2\alpha/z} f(L/t^{1/z}), \\ &= L^{2\alpha} F(t/L^z). \end{aligned} \tag{2}$$

These scaling forms are valid if dynamic scaling exists in the system with universal scaling functions, $f(X)$ and $F(x)$, where X, x are the corresponding scaled variables, $X=L/t^{1/z}$ and $x=t/L^z$. Using Eq. (2), one can test data collapsing to verify the dynamic scaling theory.

However, in some cases, surface fluctuations are not good enough to analyze kinetic roughening due to anomalous scaling that makes global roughness differ from local one. Under such circumstances, the second-order moment of height-height correlation function at the same time is employed to investigate the local roughness:

$$\begin{aligned} C_2(r,t:L) &\equiv \frac{1}{L-r}\sum_{i=1}^{L-r}[h(i+r,t)-h(i,t)]^2 \\ &= b^{2\alpha_\ell} C_2(b^{-1}r, b^{-z_\ell}t:L), \end{aligned} \tag{3}$$

where α_ℓ is the local roughness exponent and z_ℓ is the local dynamic exponent. Equation (3) can be also rewritten as if Eq. (2) with corresponding scaling functions, $g(x)$ and $G(\tau)$:

$$\begin{aligned} C_2(r,t:L) &= t^{2\alpha_\ell/z_\ell} g(r/t^{1/z_\ell}), \\ &= r^{2\alpha_\ell} G(t/L^{z_\ell}). \end{aligned} \tag{4}$$

Due to the role of active ends in polymerization and those shadowing effect [4,5], the 1D VDP growth exhibits $\alpha > \alpha_\ell$ and $z < z_\ell$, where the difference gets larger as the ratio between the diffusion rate to the deposition flux increases. The detailed numerical test will be discussed in the next section.

2. Phase Synchronization of Coupled Oscillators

Consider the original Kuramoto-type globally coupled oscillators with quenched natural frequencies that follows the unimodal distribution [6,7]:

$$\dot{\phi}_j(t) = \omega_j + \frac{K}{N}\sum_{m=1}^{N}\sin(\phi_m(t)-\phi_j(t)), \tag{5}$$

where $\phi_j(t)$ is the phase of the j-th oscillator at time t ($j,m=1,...,N$ for total number of N oscillators), and ω_j is its natural frequency, obtained from a normalized distribution $g(\omega)$. Here, we choose $g(\omega)=\frac{1}{\sqrt{2\pi}}\exp(-\frac{\omega^2}{2})$ as the unimodal function. It is noted that the functional shape of $g(\omega)$ plays a crucial role in transition nature [8].

For various N-oscillator network topologies,

$$\dot{\phi}_j(t) = \omega_j + K\sum_{m=1}^{N} A_{jm}\sin(\phi_m(t)-\phi_j(t)),$$

where A_{jm} is the adjacency matrix ($N\times N$) and results in $A_{jm}=1$ if two oscillators are connected; 0 otherwise. As

the coupling strength K increases, phase synchronization occurs at $K_c = \frac{2}{\pi g(0)}$, which can be quantified by a global complex-valued order parameter:

$$r(t)e^{i\psi(t)} \equiv \frac{1}{N}\sum_{j=1}^{N} e^{i\phi_j(t)}. \quad (6)$$

Assuming that coupled oscillators exhibit self-similar dynamics at the criticality, dynamic scaling of synchronization can be tested. We here check two initial setups: the fully synchronized state ($\phi_j(0) = 0$; $r(0) = 1$) versus the fully desynchronized state ($\phi_j(0) \in [0, 2\pi]$; $r(0) \sim 1/\sqrt{N}$) [12]. For a given value of K, r evolves either exponentially or algebraically up to its relaxation time $t_{\rm sat}$ that is also subject to the system size. In the vicinity of K_c, where $\epsilon(\equiv \frac{K-K_c}{K_c})$ is small, both the correlation volume ξ_v and time τ becomes very large (compared to the subcritical regime $K < K_c$ and the supercritical regime $K > K_c$) and algebraically decay as $\xi_v \sim \epsilon^{-\bar{\nu}}$ and $\tau \sim \epsilon^{\nu_{||}}$, respectively. However, $\xi_v \to N$ in finite systems at K_c. As a result, $\tau \sim N^{\bar{z}}$ with $\bar{z} = \nu_{||}/\bar{\nu}$. Based on the FSS theory and the thermodynamic limiting results ($N \to \infty$): $t_{\rm sat} \sim \epsilon^{-\nu_{||}}$ and $r_{\rm sat} \sim \epsilon^{\beta}$, the FSS form can be extended near and at K_c as follows:

$$r(t, N, \epsilon) = b^{-\alpha_r} r(b^{-\bar{z}}t, b^{-1}N, b^{1/\bar{\nu}}\epsilon), \quad (7)$$

where $\alpha_r = \beta/\bar{\nu}$. In the steady-state limit ($t \to \infty$), Eq. (7) is exactly the same as the earlier FSS form [9–11].

As a result, at the criticality ($\epsilon = 0$), Eq. (7) can be rewritten as the dynamic scaling form:

$$r(t, N) = t^{-\alpha_r/\bar{z}} f(t/N^{\bar{z}})$$
$$\Rightarrow t^{-\alpha_r/\bar{z}} \text{ for } t \ll N^{\bar{z}}; N^{-\alpha_r} \text{ for } t \gg N^{\bar{z}}, \quad (8)$$

where $f(x)$ is constant for $x \ll 1$ in the true scaling regime after the transient regime ($t < t_\times$ when the initial condition effect exists), and $f(x) \sim x^{\alpha_r/\bar{z}}$ for $x \gg 1$ in the saturation regime ($t \gg N^{\bar{z}}$ when N-dependency only exists). However, we note that the scaling form of Eq. (8) is valid only for an initial configuration starting at the fully synchronized state without no additional finite effect. If one chooses an initial configuration starting at the fully desynchronized state, Eq. (8) is modified due to N-dependent trivial offset and the trivial temporal scaling as the following form:

$$r_\uparrow(t, N) = N^{-1/2} t^{\theta} F(t/N^{\bar{z}})$$
$$\Rightarrow N^{-1/2}t^{\theta} \text{ for } t_\times \ll t \ll N^{\bar{z}}; N^{-\alpha_r} \text{ for } t \gg N^{\bar{z}}, \quad (9)$$

where $F(x)$ is constant for $x_*(\equiv t_\times/N^{\bar{z}}) \ll x \ll 1$ in the true scaling regime, and $F(x) \sim x^{(\alpha_r-\theta)/\bar{z}}$ for $x \gg 1$ in the saturation regime, where the order parameter exhibits very long transient trivial scaling due to random initial phases as $r_\uparrow(t) \sim N^{-1/2}t^{1/2}$. Such a behavior lasts up to $t \ll t_\times$ until the initial condition effect is washed out and the system exhibits its own true scaling.

Table 1. Critical behaviors of the order parameter are summarized near and at the criticality using Eq. (7).

	(i) $t \to \infty$	(ii) $N \to \infty$
	$r(N, \epsilon) = N^{-\beta/\bar{\nu}} F(\epsilon N^{1/\bar{\nu}})$	$r^\infty(t, \epsilon) = t^{-\beta/\nu_{||}} G(\epsilon^{\nu_{||}} t)$
$\epsilon < 0$	$r_-(N, \epsilon) \sim N^{-1/2}\epsilon^{\beta-\bar{\nu}/2}$	$r^\infty_-(t, \epsilon) \sim \exp(-bt)$
$\epsilon = 0$	$r_{\rm sat}(N) \sim N^{-\beta/\bar{\nu}}$	$r^\infty(t) \sim t^{-\beta/\nu_{||}}$
$\epsilon > 0$	$r_+(N, \epsilon) \sim \epsilon^{\beta}$	$r^\infty_+(t, \epsilon) \to [1 - A\exp(-at)]$

To resolve the universality issue, one needs to find the location of $t_\times$ accurately as well as its scaling behavior. This is definitely not a easy task and sometimes extremely tricky if the window of two consecutive scaling regimes is very narrow and one scaling interferes with the other one. Using dynamical scaling, we discuss this issue with the dynamic exponent in the true scaling regime.

Before moving onto numerical tests with the detailed discussion about the universality class of synchronization, we provide a summarized table for the extended FSS form of synchronization (see Table 1).

III. NUMERICAL RESULTS

1. The 1D VDP model

We check dynamic scaling of kinetic roughening as we numerically revisit the 1D VDP model, which consists of monomer deposition with cosine flux, surface diffusion with the relative ratio G to flux, nucleation, propagation, and coalescence process (see [4, 5] for the detail descriptions of model dynamics).

Figure 1 represents dynamic scaling of 1D VDP growth with surface fluctuations and the q-th order moment

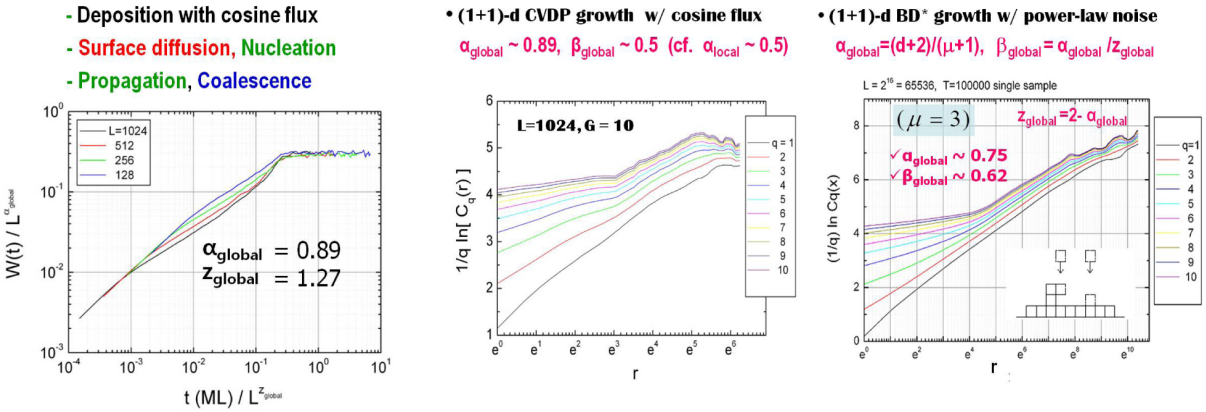

Fig. 1. (Color online) Dynamic scaling of 1D VDP growth: surface fluctuations ($W/L^{\alpha_{\rm global}}$ versus $t/L^{z_{\rm global}}$) at $G = 10$ and q-th order moment height-height correlation function in the steady-state limit, where the left panel is the VDP growth and the right one is the BD growth with the power-law decaying noise.

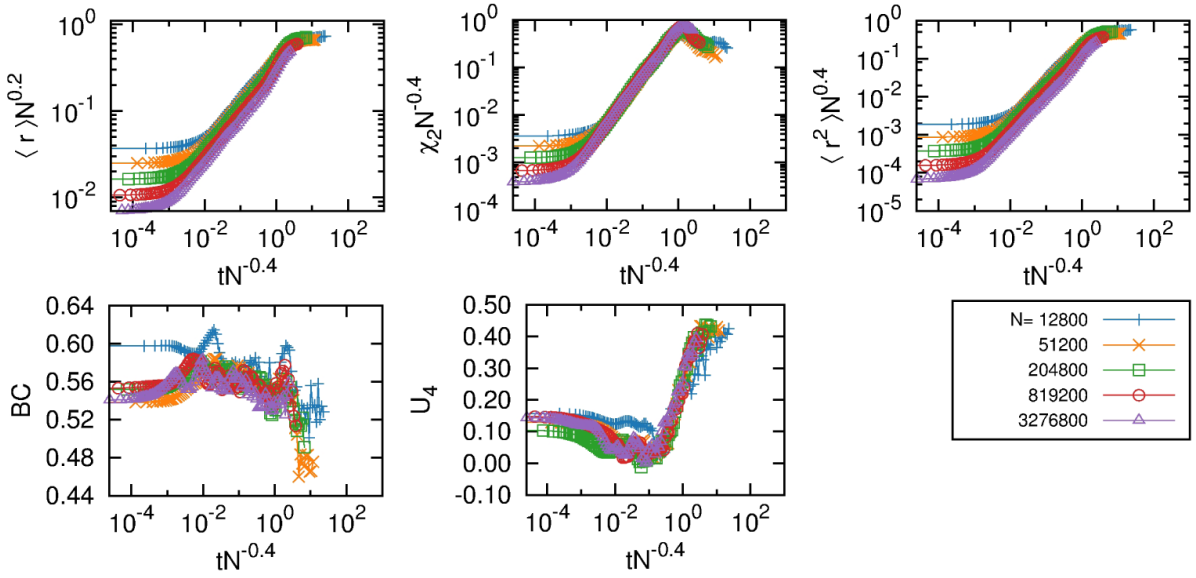

Fig. 2. (Color online) Based on dynamic scaling for the random sampling of $\{\omega_j\}$ [see Eq. (10)], numerical data collapse very well: From top (left) to bottom (right), $\langle r\rangle N^{\beta/\bar{\nu}}$, $\chi_2 N^{-\gamma/\bar{\nu}}$, $\langle r^2\rangle N^{2\beta/\bar{\nu}}$, BC, and U_4 are plotted against $tN^{-\bar{z}}$ for $N = 12800,\ 51200,\ ....,\ 204800,\ N_{\rm sample} = 2.5 \times 10^3$.

height-height correction functions that is compared to that in 1D ballistic deposition (BD) growth with power-law noise [13] in the steady-state limit.

2. Globally coupled Kuramoto-type oscillators

Dynamic scaling of synchronization is checked as we numerically revisit the Kuramoto model. We here focus on the phase synchronization transition with its critical exponents in the context of the moment analysis of r as follows:

$$\begin{aligned}
\langle r\rangle N^{\alpha_r} &= R(tN^{-z}),\\
\chi_2(\equiv N\sigma^2)N^{-\gamma/\bar{\nu}} &= X(tN^{-z}),\\
\langle r^2(N,t)\rangle N^{2\alpha_r} &= R_2(tN^{-z}),\\
{\rm BC}\Big(\equiv \frac{\langle(\tilde{r}/\sigma)^3\rangle^2+1}{\langle(\tilde{r}/\sigma)^4\rangle}\Big) &= B(tN^{-z}),\\
U_4\Big(\equiv 1-\frac{\langle\tilde{r}^4\rangle}{3\langle\tilde{r}^2\rangle^2}\Big) &= U(tN^{-z}),
\end{aligned} \qquad (10)$$

where $\tilde{r} = r - \langle r\rangle$ and $\sigma = \langle\tilde{r}^2\rangle$. It is noted that $\langle\cdots\rangle$ is the ensemble average.

Our findings of critical exponents are summarized in

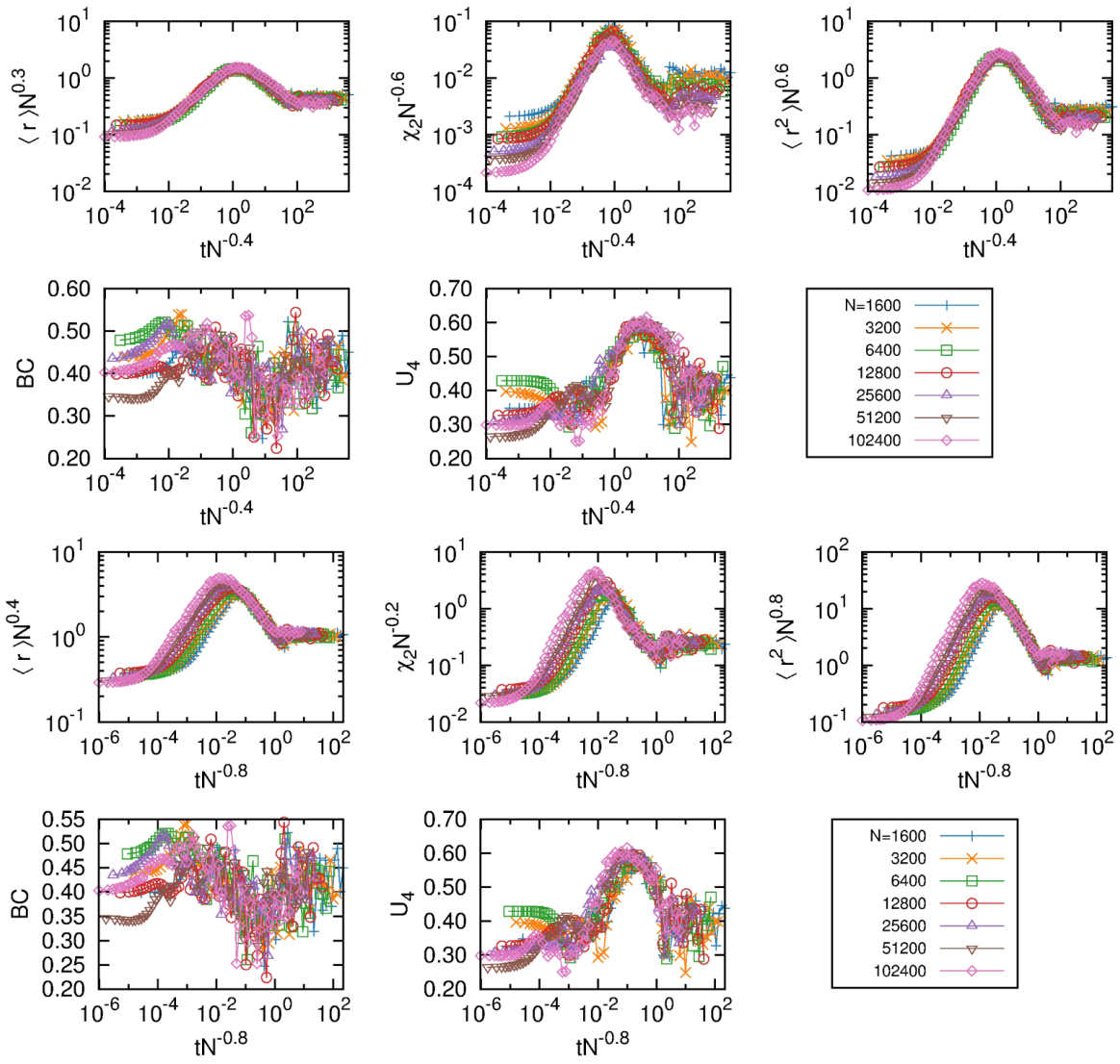

Fig. 3. (Color online) Based on dynamic scaling for the regular sampling of natural frequencies, where two scaling regimes exists, numerical data collapsed very well. The upper set is for the initial part and the lower set is for the last part: From left(top) to right(bottom), $\langle r\rangle N^{\beta/\bar{\nu}}$, $\chi_2 N^{-\gamma/\bar{\nu}}$, $\langle r^2\rangle N^{2\beta/\bar{\nu}}$, BC, and U_4 are plotted against $tN^{-\bar{z}}$ for $N = 12800,\ 51200,\ ....,\ 204800,\ N_{\rm sample} = 10^2$.

Table 2 and numerical tests are presented with the set of scaling plots in terms of Eq. (10). In Fig. 2, we test dynamic scaling for the random sampling of natural frequencies. In Fig. 3, we test dynamic scaling for the regular sampling of natural frequencies in two different scaling regimes, respectively.

Table 2. Possible universality classes of synchronization are summarized for globally coupled oscillators, where $\theta(t) = 1/2$ for $t < t_{p_1} (\sim N^{\bar{z}_1})$ with $\bar{z}_1 = 2/5$ and -1/2 for $t_{p_1} < t < t_{p_2} (\sim N^{\bar{z}_2})$ with $\bar{z}_2 = 4/5$.

	static	dynamic
$g(\omega)$	$(\beta/\bar{\nu},\ 1/\bar{\nu})$	$(\beta/\nu_{\|\|},\ \theta,\ \bar{z})$
random	(1/5, 2/5)	(1/2, 3/4, 2/5)
regular	(2/5, 4/5)	(1/2, $\theta(t)$, $z(t)$)

IV. SUMMARY WITH REMARKS

In summary, we have systematically explored dynamic scaling of 1D VDP growth and phase synchronization in the Kuramoto model for various physical quantities, and investigated scaling relations between our results and the earlier FSS ones. The merit of our work with dynamic

scaling analysis is to provide another comprehensive view of scaling properties in the context of the temporal behavior of the major physical quantities before the system reaches the steady state. This offers a guideline how to discuss and analyze critical behaviors in finite systems without the steady-state limiting results, *e.g.*, the recent work of the Kuramoto oscillators [14]. It is believed that dynamic scaling gives us essential information on analyzing scaling properties in many real systems that are often quite small in size, just as dynamic scaling of surface growth and absorbing phase transitions.

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