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The zeta functions of Ruelle and Selberg for hyperbolic manifolds with cusps

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Abstract In this paper, we study the Ruelle zeta function and the Selberg zeta functions attached to the fundamental representations for real hyperbolic manifolds with cusps. In particular, we show that they have meromorphic extensions to $\mathbb{C}$ and satisfy functional equations. We also derive the order of the singularity of the Ruelle zeta function at the origin. To prove these results, we completely analyze the weighted unipotent orbital integrals on the geometric side of the Selberg trace formula when test functions are defined for the fundamental representations.

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1 Introduction

In his seminal paper [8], Fried extensively studied the zeta functions of Ruelle and Selberg for certain compact manifolds. In particular, he showed that these zeta functions have meromorphic extensions to the whole complex plane $\mathbb{C}$ although these are a priori defined over a right half plane. The method Fried employed is the symbolic dynamics of Axiom A flows developed in [2,35], which can be applied to the case of the convex cocompact hyperbolic manifolds (see the paper of Patterson-Perry [32]). Combining these results and the Selberg trace formula, he also derived the order of the

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singularity of the Ruelle zeta function at the origin and the relation of the coefficient of its leading term with the Reidemeister torsion for compact hyperbolic manifolds in [9].

The main purpose of this paper is to extend some results of Fried [8,9] to noncompact hyperbolic manifolds with cusps. We show that the Ruelle zeta function and the Selberg zeta functions attached to the fundamental representations have meromorphic extensions to the whole complex plane $\mathbb{C}$ and satisfy the functional equations. We also derive the exact value of the order of the singularity of the Ruelle zeta function at the origin. The relationship of the coefficient of the leading term of the Ruelle zeta function at the origin with the analytic torsion is given in the companion papers [30,31].

One of motivations of these works is to resolve the geometric analogues of some conjectures raised in [6,24] for the Hasse-Weil zeta function. A study of this direction about the geometric analogues of these conjectures was also given in [39] for compact hyperbolic manifolds of dimension 3. This paper and [30,31] resolve these problems for noncompact hyperbolic manifolds with cusps.

Let us explain our results in more detail. The real hyperbolic space of dimension d can be realized as the symmetric space G/K, where $G = \mathrm{SO}_0(d,1)$, $K = \mathrm{SO}(d)$. Let us denote by $\Gamma \subset G$ a discrete subgroup of G which is torsion free and $\mathrm{Vol}(\Gamma\backslash G) < \infty$, where the notation Vol denotes the volume with respect to the measure in (2.3). Throughout this paper, the measure in (2.3) and the induced metric from it are normalized so that the symmetric space G/K has the constant curvature (-1). Now let us denote by $\mathfrak{P}_\Gamma$ the set of Γ-conjugacy classes of Γ-cuspidal parabolic subgroups in G. We also assume that the discrete subgroup Γ satisfies the condition

$$\Gamma_P := \Gamma \cap P = \Gamma \cap N(P) \quad \text{for} \quad P \in \mathfrak{P}_\Gamma, \tag{1.1}$$

where $N(P)$ denotes the unipotent radical of P. Note that this condition is satisfied when Γ is neat, that is, the group generated by the eigenvalues of Γ contains no root of unity. Although this condition is rather restrictive, we assume it to avoid complicatedness for general case as in [4,11,28,37,42].

Now the resulting manifold $X_\Gamma = \Gamma\backslash G/K$ is a noncompact hyperbolic manifold with cusps. The Ruelle zeta function for X_Γ is defined by

$$R_\chi(s) := \prod_{\{\gamma\}_\Gamma \in \mathrm{P}\Gamma_{\mathrm{hyp}}} \det\left(\mathrm{Id} - \chi(\gamma) e^{-s\, l(C_\gamma)}\right)^{-1}$$

for $\mathrm{Re}(s) > (d-1)$. Here $\mathrm{P}\Gamma_{\mathrm{hyp}}$ denotes the set of Γ-conjugacy classes $\{\gamma\}_\Gamma$ of the primitive hyperbolic elements γ in Γ, (χ, V_χ) denotes a unitary representation of Γ so that det is taken over V_χ, and $l(C_\gamma)$ denotes the length of the closed geodesic C_γ determined by a hyperbolic element γ, where the length $l(C_\gamma)$ is measured by the normalized metric. The following theorem is the main result for the Ruelle zeta function $R_\chi(s)$.

Theorem 1.1 *For the co-finite hyperbolic manifold X_Γ with cusps, the Ruelle zeta function $R_\chi(s)$ a priori defined for* $\mathrm{Re}(s) > (d-1)$ *has a meromorphic extension to* *the complex plane* $\mathbb{C}$. *In addition, the Ruelle zeta function* $R_\chi(s)$ *satisfies the following functional equation*

$$\begin{aligned} R_\chi(-s)\, R_\chi(s)^{-1} \;&=\; \exp\left((-1)^{n-1}\, 4(n+1)\binom{2n}{n}^{-1} s\right)^{\dim V_\chi \mathrm{Vol}(\Gamma\backslash G)} \\ &\quad\cdot C_\chi(d,s)\; Y(d,s)^{d_c(\chi)}\; \exp\left(-Q_\chi(d,s)\right) \qquad \text{if } d = 2n+1, \\ R_\chi(-s)\, R_\chi(s) \;&=\; (-1)^{n \dim V_\chi\, E(X_\Gamma)}\, (2\sin(\pi s))^{d\, \dim V_\chi\, E(X_\Gamma)} \\ &\quad\cdot C_\chi(d,s)\; Y(d,s)^{d_c(\chi)}\; \exp\left(-Q_\chi(d,s)\right) \qquad \text{if } d = 2n. \end{aligned}$$

Here $C_\chi(d,s)$ *is a meromorphic function defined by scattering operators* $C_\chi^k(\sigma_k, s)$*'s,* $Y(d,s)$ *is a rational function,* $d_c(\chi)$ *is a non-negative integer given in* (2.16)*,* $Q_\chi(d,s)$ *is a polynomial, and* $E(X_\Gamma)$ *denotes the Euler characteristic of* X_Γ*.*

The precise definitions of $C_\chi(d,s)$, $Y(d,s)$, $Q_\chi(d,s)$ are given in Theorems 5.2 and 5.3. If $d = 2n$, the Euler characteristic $E(X_\Gamma)$ is a rational multiple of $(-1)^n$ $\mathrm{Vol}(\Gamma\backslash G)$ by Proposition 4.9 under our normalization.

The results in Theorem 1.1 for the odd dimensional case were announced in [13]. We also remark that Theorem 1.1 have been proved in [3,12,20] assuming that X_Γ is compact.

To prove Theorem 1.1, we use the usual relation of the Ruelle zeta function $R_\chi(s)$ with the Selberg zeta functions $Z_\chi(\sigma_k, s)$'s attached to the representation σ_k of $M = \mathrm{SO}(d-1) \subset K$ acting on $\wedge^k(\mathbb{C}^{d-1})$. Then the results in Theorem 1.1 will follow from the corresponding results for the $Z_\chi(\sigma_k, s)$'s (see Theorems 4.6 and 4.15). To prove these we follow the traditional way to use the Selberg trace formula as in Gangolli and Warner [11]. However, along this approach we have to compute the weighted unipotent orbital integrals on the geometric side which have not been fully analyzed for nontrivial K-types. Applying the result in [16], we completely analyze these terms for our cases. These terms for the spinor bundle case were explicitly computed in [29] and we continue this study for the bundle of k-forms in this paper. We also use the multiple sine function introduced by Kurokawa [23] to deal with the Plancherel measures appearing in the identity orbital integral for the even dimensional case, which is useful in the derivation of the functional equations of $R_\chi(s)$, $Z_\chi(\sigma_k, s)$.

Our approach following Gangolli and Warner [11] seems to be traditional comparing the approach in [3] employing analysis of differential operators. But a generalization of the method in [3] to hyperbolic manifolds with cusps involves several nontrivial analytic problems since we have to deal with continuous spectrum of the differential operators. For instance, we need some regularization to consider the distributional trace of the wave operator of the Laplacian for hyperbolic manifolds with cusps as in [15]. An extension of the method in [3,15] to hyperbolic manifolds with cusps seems to be highly interesting and worthy of extensive study, but it is beyond the scope of this paper.

As a byproduct of the proof of Theorem 1.1, we can describe the locations of the zeros and poles of $Z_\chi(\sigma_k, s)$ explicitly in terms of the spectral data of the Laplacian Δ_k acting on spaces of k-forms twisted by χ (see Theorem 4.6). Using this we can obtain an explicit formula of the order N_0 of the singularity of $R_\chi(s)$ at $s = 0$. Here N_0 denotes an integer such that $\lim_{s\to 0} s^{N_0} R_\chi(s)$ is a nonzero finite value.

Theorem 1.2 *The following equalities hold,*

$$N_0 = 2\sum_{k=0}^{n}(-1)^k(n+1-k)\beta_k + \sum_{k=0}^{n-1}(-1)^{k+1}b_k\binom{2n}{k} + d_c(\chi)\,(-1)^n\binom{2n-2}{n-1} \qquad \text{if } d = 2n+1,$$

$$N_0 = -n\dim V_\chi\, E(X_\Gamma) + \sum_{k=0}^{n-1}(-1)^{k+1}b_k\binom{2n-1}{k} + d_c(\chi)\,(-1)^{n-1}\binom{2n-3}{n-2} \qquad \text{if } d = 2n,$$

where $\beta_k := \dim\ker(\Delta_k)$ *and* b_k *is the order of the singularity of* $\det C^k_\chi(\sigma_k, s)$ *at* $s = \frac{d-1}{2} - k$.

This result for the odd dimensional case was announced in [13]. Theorem 1.2 is a generalization of Corollary 1 and Theorem 3 in [19] and Theorem 3 in [9] to the case of a noncompact hyperbolic manifold with cusps, where the second term (the scattering contribution) and the third term (the cuspidal contribution from the unipotent term) appear. Theorem 1.2 is an important ingredient in the study of the analytic torsion for hyperbolic manifolds with cusps [30,31].

The structure of this paper is as follows: In Sect. 2, we review the basics of the Selberg trace formula which we use as a main tool. In Sect. 3, we completely analyze the weighted unipotent orbital integral appearing on the geometric side of the Selberg trace formula for our case. In Sect. 4, we prove meromorphic extensions and functional equations of the Selberg zeta functions $Z_\chi(\sigma_k, s)$'s. In Sect. 5, we prove Theorems 1.1 and 1.2 using the results proved in Sect. 4.

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2 Selberg trace formula

2.1 Notation and normalization

We denote the Lie algebras of G, K by $\mathfrak{g} = \mathfrak{so}(d, 1)$, $\mathfrak{k} \cong \mathfrak{so}(d)$ respectively. The Cartan involution θ on $\mathfrak{g}$ gives us the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}, \mathfrak{p}$ are the $1, -1$ eigenspaces of θ respectively. The invariant metric of constant curvature (-1) on G/K corresponds to the normalized Cartan-Killing form

$$\langle X, Y\rangle := -\frac{1}{2d-2}C(X, \theta Y), \tag{2.1}$$

where the Killing form is defined by $C(X, Y) = \mathrm{Tr}(\mathrm{ad}\, X \circ \mathrm{ad}\, Y)$ for $X, Y \in \mathfrak{g}$.

Let $\mathfrak{a}$ be a fixed maximal abelian subspace of $\mathfrak{p}$. Then the dimension of $\mathfrak{a}$ is one. Let $M \cong \mathrm{SO}(d-1)$ be the centralizer of $A = \exp(\mathfrak{a})$ in K with Lie algebra $\mathfrak{m}$. We put β to be the positive restricted root of $(\mathfrak{g}, \mathfrak{a})$ with $||\beta|| = 1$, where the norm $||\cdot||$ is induced from (2.1). Let ρ denote the half sum of the positive roots of $(\mathfrak{g}, \mathfrak{a})$, that is, $\rho = \frac{(d-1)}{2}\beta$. Later on, we shall use the identification

$$\mathfrak{a}_{\mathbb{C}}^* \cong \mathbb{C} \quad \text{by} \quad \lambda\beta \to \lambda. \tag{2.2}$$

Let $\mathfrak{n}$ be the positive root space of β and $N = \exp(\mathfrak{n}) \subset G$. The Iwasawa decomposition is given by $G = NAK$. From now on we fix the following Haar measure on G,

$$dg = a^{-2\rho} dn\, da\, dk, \tag{2.3}$$

where $g = nak$ is the Iwasawa decomposition and $a^{-2\rho} = \exp(-2\rho(\log a))$. Here dk is the Haar measure on K with $\int_K dk = 1$, da is the Euclidean Lebesgue measure on A given by the identification $A \cong \mathbb{R}$ via $a = \exp(tH)$ with $H \in \mathfrak{a}$, $\beta(H) = 1$, and dn is the Euclidean Lebesgue measure on N induced by the normalized Cartan-Killing form $\langle\cdot,\cdot\rangle$ given in (2.1).

Let us denote the irreducible fundamental representation of $M = \mathrm{SO}(d-1)$ acting on $\wedge^\ell(\mathbb{C}^{d-1})$ by σ_ℓ if $d = 2n$ or if $d = 2n+1$ with $\ell \neq n$. When $d = 2n+1$ and $\ell = n$, there are two irreducible half spin representations acting on $\wedge^n(\mathbb{C}^{2n})$ denoted by $\sigma_n^\pm$. We denote by $d(\sigma_\ell)$ the dimension of representation space of σ_ℓ unless $d = 2n+1$ and $\ell = n$, and by $d(\sigma_n)$ the corresponding one of $\sigma_n^\pm$ if $d = 2n+1$. We also denote the irreducible fundamental representation of $K = \mathrm{SO}(d)$ acting on $\wedge^k(\mathbb{C}^d)$ by τ_k if $d = 2n+1$ or $d = 2n$ with $k \neq n$, by $\tau_n^\pm$ if $d = 2n$. These representations of K satisfy the following branching laws:

(1) For $k \neq n$ with $d = 2n$ or $d = 2n+1$,

$$[\tau_k|_M : \sigma_\ell] = 1 \quad \text{if and only if} \quad \sigma_\ell = \sigma_k \quad \text{or} \quad \sigma_\ell = \sigma_{k-1},$$

(2) For $k = n$ and $d = 2n$,

$$[\tau_n^\pm|_M : \sigma_\ell] = 1 \quad \text{if and only if} \quad \sigma_\ell = \sigma_n,$$

(3) For $k = n$ and $d = 2n+1$,

$$[\tau_n|_M : \sigma_\ell] = 1 \quad \text{if and only if} \quad \sigma_\ell = \sigma_{n-1} \quad \text{or} \quad \sigma_\ell = \sigma_n^\pm.$$

For other cases than listed above, $[\tau_k|_M : \sigma_\ell] = 0$.

2.2 Selberg trace formula

Let us choose a unitary representation χ of Γ on a finite dimensional hermitian vector space V_χ. We now consider the right quasi-regular representation R_χ on

$$\mathfrak{H}_\chi := \{\phi : G \to V_\chi | \phi(\gamma x) = \chi(\gamma)\phi(x) \quad \text{for } \gamma \in \Gamma, x \in G, \quad ||\phi|| \in L^2(\Gamma\backslash G)\}$$

given by $(R_\chi(x)\phi)(y) = \phi(yx)$. It is known that this representation R_χ of G on $\mathfrak{H}_\chi$ decomposes into a discrete part and a continuous part. That is,

$$R_\chi = R_\chi^d \oplus R_\chi^c \quad \text{acts on} \quad \mathfrak{H}_\chi = \mathfrak{H}_\chi^d \oplus \mathfrak{H}_\chi^c.$$

The action R_χ^d on $\mathfrak{H}_\chi^d$ is a direct sum of irreducible representations, each of them occurring with finite multiplicity and the action of R_χ^c on $\mathfrak{H}_\chi^c$ is a direct integral, with no irreducible subrepresentations, of principal series.

Let us denote by $\mathcal{C}^p(G)$ the Harish-Chandra L^p-Schwartz space over G. Here $\mathcal{C}^p(G)$ is the space of all functions $f \in C^\infty(G)$ such that

$$\sup_{g\in G} (1+\sigma(g))^m \Psi(g)^{-\frac{2}{p}} |D_1 D_2 f(g)| < \infty \quad \text{for any } m \geq 0, \;\; D_1, D_2,$$

where $\sigma(g)$ is the geodesic distance between the cosets eK and gK in G/K,

$$\Psi(g) = \int_K e^{-\rho(H(gk))}\, dk$$

for the Iwasawa decomposition $gk = K(gk)\exp(H(gk))N(gk)$, and D_1, D_2 denote the right, left invariant differential operators, respectively. For a test function $h \in \mathcal{C}^p(G)$ with $0 < p < 1$, which is right K-finite, the induced operator $R_\chi^d(h)$ is of trace class and

$$\mathrm{Tr}\, R_\chi^d(h) = \mathrm{Tr} \int_G h(g)\, R_\chi^d(g)\, dg = \sum_{\pi\in\widehat{G}} m_\chi(\pi)\, \mathrm{Tr}\, \pi(h), \tag{2.4}$$

where $m_\chi(\pi)$ denotes the multiplicity of $\pi \in \widehat{G}$ in $\mathfrak{H}_\chi^d$. Now the Selberg trace formula applied to $h \in \mathcal{C}^p(G)$ with $0 < p < 1$ has the following form,

$$\mathrm{Tr}\, R_\chi^d(h) = I_\chi(h) + H_\chi(h) + U_\chi(h) + W_\chi(h) + S_\chi(h) + J_\chi(h). \tag{2.5}$$

We refer to Theorem 6.3 in [42] for the equality (2.5) when χ is trivial. When χ is nontrivial, it also can be derived in a similar way as in [42] combining the result in Sect. 3 of [31]. Here I_χ, H_χ, U_χ are given by the identity, hyperbolic, unipotent orbital integrals respectively. These orbital integrals are invariant tempered distributions on G which were fully analyzed in [36]. These terms will be discussed in this section. The next term W_χ is given by weighted unipotent orbital integrals, which are not invariant and have been the main difficulty in the application of the Selberg trace formula for hyperbolic manifolds with cusps. This term will be analyzed in the next section by applying the result in [16]. The other two spectral terms S_χ, J_χ are called scattering and residual terms respectively.

2.3 Identity orbital integral

Let us recall

$$I_\chi(h) = \dim V_\chi \cdot \mathrm{Vol}(\Gamma\backslash G) \cdot h(e).$$

By the Plancherel theorem,

$$h(e) = \sum_{\omega\in\widehat{G}_d} d(\omega)\Theta_\omega(h) + \sum_{\sigma\in\widehat{M}} \frac{1}{4\pi}\int_{-\infty}^{\infty} \Theta_{\sigma,\lambda}(h)\,\mu(\sigma,\lambda)\,d\lambda, \tag{2.6}$$

where $d(\omega)$ denotes the formal degree of $\omega\in\widehat{G}_d$ and $\mu(\sigma,\lambda)$ denotes the Plancherel measure corresponding to the unitary principal series $\pi_{\sigma,\lambda}$. Here $\pi_{\sigma,\lambda} = \mathrm{Ind}_{MAN}^{G}(\sigma\otimes a^{i\lambda+\rho}\otimes 1_N)$ is the (non-unitary) principal series representation of G for $(\sigma,\lambda)\in\widehat{M}\times\mathfrak{a}^*_{\mathbb{C}}$ and $\Theta_{\sigma,\lambda}(h) = \mathrm{Tr}\,\pi_{\sigma,\lambda}(h)$. By Theorem 3.1 in [25] (we use a different normalization of Haar measure on G from [25]), the Plancherel measure corresponding to $\pi_{\sigma_k,\lambda}$, under our normalization, is given by

$$\begin{aligned}
\mu(\sigma_k,\lambda) &= \frac{\pi}{2^{4n-2}\,\Gamma(n+1/2)^2}\cdot d(\sigma_k)\cdot p(k,\lambda)\\
&\quad\text{with}\quad p(k,\lambda) = \prod_{j=1}^{k}\left(\lambda^2+(n-j+1)^2\right)\prod_{j=k+1}^{n}\left(\lambda^2+(n-j)^2\right)\\
&\quad\text{if}\ d = 2n+1,\\
\mu(\sigma_k,\lambda) &= \frac{\pi}{2^{4n-4}\,\Gamma(n)^2}\cdot d(\sigma_k)\cdot p(k,\lambda)\cdot\tanh(\pi\lambda)\\
&\quad\text{with}\quad p(k,\lambda) = \lambda\prod_{j=1}^{k}\left(\lambda^2+\left(n-j+\frac{1}{2}\right)^2\right)\\
&\qquad\times\prod_{j=k+1}^{n-1}\left(\lambda^2+\left(n-j-\frac{1}{2}\right)^2\right)\quad\text{if}\ d=2n,
\end{aligned}\tag{2.7}$$

where σ_n means $\sigma_n^{\pm}$ when $d = 2n+1$.

Let us recall that for $G = \mathrm{SO}_0(2n+1,1)$ there is no discrete series so that there are no terms from $\widehat{G}_d$ on the right hand side of (2.6). For $G = \mathrm{SO}_0(2n,1)$, there is a discrete series, hence the contribution from $\widehat{G}_d$ is nontrivial in general. But, it is trivial for a test function $h = \mathrm{tr}(f^k)$, where $f^k : G\to \mathrm{End}(V_{\tau_k})$ is a τ_k-radial function constructed in Sect. 4.1 by Theorems 3.2, 6.5, 6.12, and 6.14 in [34].

2.4 Hyperbolic orbital integral

The term $H_\chi(h)$ is given by hyperbolic orbital integrals as follows,

$$H_\chi(h) = \sum_{\{\gamma\}_\Gamma\in\Gamma_{\mathrm{hyp}}} \mathrm{tr}\,\chi(\gamma)\cdot\mathrm{Vol}(\Gamma_\gamma\backslash G_\gamma)\cdot\int_{G_\gamma\backslash G} h(g^{-1}\gamma g)\,d(G_\gamma g), \tag{2.8}$$

where Γ_{hyp} denotes the set of the Γ-conjugacy classes $\{\gamma\}_\Gamma$ of the hyperbolic elements γ in Γ. Here the measure $d(G_\gamma g)$ on $G_\gamma \backslash G$ is induced from dg in (2.3) and normalized such that

$$\int_G \psi(g)\, dg = \int_{G_\gamma \backslash G} \int_{G_\gamma} \psi(xg)\, dx\, d(G_\gamma g)$$

for $\psi \in C_0^\infty(G)$. For the hyperbolic orbital integral, we may assume that a hyperbolic element $\gamma \in \Gamma$ has the form $m_\gamma a_\gamma \in MA^+$, where $A^+ = \{a \in A \,|\, a = \exp(tH),\ t > 0\}$. By Section 6 in [41], we also have

$$\begin{aligned}
&\mathrm{Vol}(\Gamma_\gamma \backslash G_\gamma) \cdot \int_{G_\gamma \backslash G} h(g^{-1}\gamma g)\, d(G_\gamma g) \\
&\quad = \sum_{\sigma \in \widehat{M}} l(C_\gamma) j(\gamma)^{-1} D(\gamma)^{-1}\, \overline{\mathrm{tr}\,\sigma(m_\gamma)}\, \frac{1}{2\pi} \int_{-\infty}^{\infty} \Theta_{\sigma,\lambda}(h) e^{-il(C_\gamma)\lambda}\, d\lambda, \qquad (2.9)
\end{aligned}$$

where $l(C_\gamma)$ denotes the length of the closed geodesic determined by γ, $j(\gamma)$ denotes the positive integer such that $\gamma = \gamma_0^{j(\gamma)}$ with a primitive γ_0 and $D(\gamma)$ is given by

$$D(\gamma) = D(m_\gamma a_\gamma) = e^{\frac{d-1}{2} l(C_\gamma)} \left| \det \left(\mathrm{Ad}(m_\gamma a_\gamma)^{-1} - \mathrm{Id}|_{\mathfrak{n}} \right) \right|.$$

2.5 Unipotent terms

By the computation in [28], under our normalization the terms $U_\chi(h)$ and $W_\chi(h)$ are given by the sum of the following terms

$$\mathrm{Vol}(\Gamma_P \backslash N(P)) \lim_{s \to 0} \frac{d}{ds} (s\, \zeta_P(s, \chi) T_P(h, s)) \qquad (2.10)$$

for $P \in \mathfrak{P}_\Gamma$. Here the Epstein type zeta function $\zeta_P(s, \chi)$ is defined by

$$\zeta_P(s, \chi) = \sum_{\eta \in \Gamma_P, \eta \neq e} \mathrm{tr}\, \chi(\eta) |X_\eta|^{-(d-1)(s+1)} \quad \text{for} \quad \mathrm{Re}(s) > 0,$$

where $\eta = \exp(X_\eta)$ and $|X_\eta|^2 = \langle X_\eta, X_\eta \rangle$. The other term $T_P(h, s)$ is given by

$$T_P(h, s) = \frac{1}{A(\mathfrak{n})} \int_N \int_K h(knk^{-1}) |\log n|^{(d-1)s}\, dk\, dn,$$

where $A(\mathfrak{n})$ is the volume of the unit sphere in $\mathfrak{n}$. By Section 1 of [28] (and Sect. 7 of [42]), we know that $s \mapsto T_P(h, s)$ is holomorphic on a certain strip containing the imaginary axis.

Now let us observe that $\chi|_{\Gamma_P}$ decomposes into one-dimensional representations χ_θ's of Γ_P (since Γ_P is abelian by (1.1)) such that

$$\chi_\theta(\eta) = \exp\left(2\pi i(n_1\theta_1 + \cdots + n_{d-1}\theta_{d-1})\right) \quad \text{for} \quad \eta = \prod_{j=1}^{d-1} \eta_j^{n_j},$$

where $\{\eta_i\}$ denotes a fixed basis of Γ_P. For $P \in \mathfrak{P}_\Gamma$, we decompose

$$V = V_P \oplus V_P{}^{\perp},$$

where $V_P \subset V$ is the maximal subspace on which $\chi|_{\Gamma_P}$ acts trivially, so that χ decomposes into a direct sum of Id_{V_P} and χ_θ's with nontrivial $\theta = (\theta_1, \ldots, \theta_{d-1})$, that is, one of θ_i is not an integer. Putting $d_P(\chi) = \dim V_P$, we have

$$\zeta_P(s,\chi) = d_P(\chi) \cdot \sum_{\eta \in \Gamma_P, \eta \neq e} |X_\eta|^{-(d-1)(s+1)} + \sum_\theta \sum_{\eta \in \Gamma_P, \eta \neq e} \chi_\theta(\eta)|X_\eta|^{-(d-1)(s+1)}, \tag{2.11}$$

where the second sum runs over the non-trivial θ. The first sum on the right side of (2.11) is given by the ordinary Epstein zeta function which has a simple pole at $s = 0$. The second sum on the right side of (2.11) is given by entire functions by Proposition 4.2 in [31]. Therefore we conclude

$$\lim_{s\to 0} \frac{d}{ds}\left(s\,\zeta_P(s,\chi)T_P(h,s)\right) = d_P(\chi)\left(C_P T_P(h) + R_P T_P'(h)\right) + \widetilde{C}_P T_P(h), \tag{2.12}$$

where C_P, R_P denote the constant term and the residue of the Epstein zeta function at $s = 0$ respectively, $\widetilde{C}_P$ denotes the sum of the constant terms of $\zeta_P(s,\chi_\theta)$ with non-trivial θ at $s = 0$, and

$$T_P(h) = \frac{1}{A(\mathfrak{n})} \int_N \int_K h(knk^{-1})\, dk\, dn,$$

$$T_P'(h) = \frac{(d-1)}{A(\mathfrak{n})} \int_N \int_K h(knk^{-1}) \log|\log n|\, dk\, dn.$$

The term $U_\chi(h)$ is given by the sum over $P \in \mathfrak{P}_\Gamma$ of

$$\mathrm{Vol}(\Gamma_P \backslash N(P))\ \left(d_P(\chi)C_P + \widetilde{C}_P\right)\ T_P(h), \tag{2.13}$$

which is the invariant part of the right side of (2.12). Moreover, we have

$$T_P(h) = \frac{1}{A(\mathfrak{n})} \sum_{\sigma \in \widehat{M}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \Theta_{\sigma,\lambda}(h)\, d\lambda. \tag{2.14}$$

The term $W_\chi(h)$ is given by the sum over $P \in \mathfrak{P}_\Gamma$ of the remaining part $d_P(\chi) R_P T'_P(h)$ on the right side of (2.12). By the computation in [5],

$$\mathrm{Vol}(\Gamma_P \backslash N(P))\, R_P \frac{(d-1)}{A(\mathfrak{n})} = 1$$

under our normalization. Hence

$$W_\chi(h) = d_c(\chi) \int_N \int_K h(knk^{-1}) \log|\log n|\, dk\, dn, \tag{2.15}$$

where

$$d_c(\chi) = \sum_{P_j \in \mathfrak{P}_\Gamma} d_{P_j}(\chi). \tag{2.16}$$

2.6 Scattering and residual terms

Let

$$L^2(M) = \sum_{\sigma \in \widehat{M}} \oplus\, d(\sigma) H_\sigma, \quad R_M = \sum_{\sigma \in \widehat{M}} \oplus\, d(\sigma)\sigma$$

be the decomposition of the right regular representation R_M of M on $L^2(M)$, where $d(\sigma) = \dim H_\sigma$. A similar induction procedure to the principal series representation starting with R_M instead of $\sigma \in \widehat{M}$ gives rise to a unitary representation of G,

$$\sum_{\sigma \in \widehat{M}} \oplus\, \pi(\sigma, \lambda) \quad \text{acts on} \quad \sum_{\sigma \in \widehat{M}} \oplus\, \mathcal{H}(\pi(\sigma, \lambda)),$$

where

$$\pi(\sigma, \lambda) = \begin{cases} d(\sigma)\pi_{\sigma,\lambda} & \text{if } w\sigma = \sigma \\ d(\sigma)\pi_{\sigma,\lambda} \oplus d(w\sigma)\pi_{w\sigma,\lambda} & \text{if } w\sigma \neq \sigma \end{cases}, \tag{2.17}$$

and $\mathcal{H}(\pi(\sigma, \lambda))$ denotes the representation space of $\pi(\sigma, \lambda)$. Here w is the nontrivial element in $W(G, A)$. Now for $P_j \in \mathfrak{P}_\Gamma$ with the corresponding decomposition $P_j = M_j A_j N_j$, where $P_j = x_j P_0 x_j^{-1}$, $N_j = x_j N x_j^{-1}$, $A_j = x_j A x_j^{-1}$, $M_j = x_j M x_j^{-1}$ for certain $x_j \in K$ and $P_0 = MAN$, the above definitions carry over to each M_j with obvious changes of notation such as $\pi(\sigma_{(j)}, \lambda_{(j)})$ for $1 \leq j \leq \kappa$ with $\kappa := |\mathfrak{P}_\Gamma|$. When $[\tau|_{M_j} : \sigma_{(j)}] \neq 0$, we put $\mathcal{H}(\sigma_{(j)}, \tau)$ to be the τ-isotypic component of $\mathcal{H}(\pi(\sigma_{(j)}, \lambda_{(j)}))$. Let us remark that $\sigma_{(j)}$ should not be confused with the fundamental representation σ_ℓ acting on $\wedge^\ell(\mathbb{C}^{d-1})$.

Now for $P = P_j \in \mathfrak{P}_\Gamma$ and $\Phi \in V_P \otimes \mathcal{H}(\sigma, \tau)$, where V_P denotes the maximal invariant subspace of V_χ under $\chi|_{\Gamma_P}$, the Eisenstein series attached to Φ is defined as

$$E(P, \Phi, s, x) := \sum_{\gamma \in \Gamma/\Gamma_P} \chi(\gamma) e^{(s+\rho)(H(\gamma^{-1}x))} \Phi(\gamma^{-1}x) \quad \text{for } \operatorname{Re}(s) > \frac{d-1}{2}, \tag{2.18}$$

where $H(x) = H_j(x)$ is given by the decomposition $x = N_j(x) \exp(H_j(x)) K(x)$. The infinite sum on the right hand side of (2.18) is absolutely and uniformly convergent on compact sets in G over the half plane $\operatorname{Re}(s) > \frac{d-1}{2}$, and $E(P, \Phi, s, x)$ extends meromorphically to $\mathbb{C}$. These facts can be proved as in [14,27]. For $P_i, P_j \in \mathfrak{P}_\Gamma$, the constant term of $E(P_i, \Phi, s, x)$ along P_j is defined by

$$E_{P_j}(P_i, \Phi, s, x) = \frac{1}{\operatorname{vol}(\Gamma \cap N_j \backslash N_j)} \int_{\Gamma \cap N_j \backslash N_j} E(P_i, \Phi, s, nx)\, dn$$

and has the following expression along P_j,

$$E_{P_j}(P_i, \Phi, s, x) = \sum_{w \in W(A_i, A_j)} e^{(ws+\rho)(H_j(x))} \left(C^\tau_{ji}(w, s)\Phi \right)(x),$$

where $W(A_i, A_j)$ denotes the set of all bijections $w : A_i \to A_j$ defined by $wa_i = xa_i x^{-1}$ for $x \in K$ and

$$C^\tau_{ji}(w, s) : V_{P_i} \otimes \mathcal{H}(\sigma_{(i)}, \tau) \longrightarrow V_{P_j} \otimes \mathcal{H}(\sigma_{(j)}, \tau), \quad w \in W(A_i, A_j).$$

Now combining the operators $C^\tau_{ji}(x_i w x_j^{-1}, x_j \cdot s)$ with the nontrivial element $w \in W(A, A)$ defines the scattering operator

$$C^\tau_\chi(\sigma, s) \quad \text{on} \quad \mathcal{H}_\chi(\sigma, \tau) := \sum_{j=1}^{\kappa} \oplus\ V_{P_j} \otimes \mathcal{H}(\sigma_{(j)}, \tau).$$

When $\tau = \tau_k$, we denote $C^{\tau_k}_\chi(\sigma, s)$ by $C^k_\chi(\sigma, s)$ for simplicity. The scattering operator has a meromorphic extension to $\mathbb{C}$ and it satisfies the well-known functional equations

$$C^\tau_\chi(\sigma, s) C^\tau_\chi(\sigma, -s) = \operatorname{Id}, \quad C^\tau_\chi(\sigma, s)^* = C^\tau_\chi(\sigma, \bar{s}). \tag{2.19}$$

Now the scattering term $S_\chi(h)$ and the residual term $J_\chi(h)$ are given by

$$S_\chi(h) = -\sum_{\sigma\in\widehat{M}}[\tau|_M:\sigma]\,\frac{1}{4\pi}\int_{-\infty}^{\infty}\operatorname{tr}\left(C_\chi^\tau(\sigma,-i\lambda)\partial_{i\lambda}C_\chi^\tau(\sigma,i\lambda)\pi_\chi(\sigma,\lambda)(h)\right)d\lambda$$
$$= -\sum_{\sigma\in\widehat{M}}[\tau|_M:\sigma]\,\frac{d(\sigma)}{4\pi}\int_{-\infty}^{\infty}\Theta_{\sigma,\lambda}(h)\operatorname{tr}\left(C_\chi^\tau(\sigma,-i\lambda)\partial_{i\lambda}C_\chi^\tau(\sigma,i\lambda)\right)d\lambda,$$
$$J_\chi(h) = \sum_{\sigma\in\widehat{M}}[\tau|_M:\sigma]\,\frac{d(\sigma)}{4}\,\Theta_{\sigma,0}(h)\operatorname{tr}\left(C_\chi^\tau(\sigma,0)\right),$$

where $\pi_\chi(\sigma,\lambda)$ is the representation of G on $\mathcal{H}_\chi(\sigma,\tau) := \sum_{j=1}^{\kappa}\oplus\, V_{P_j}\otimes\mathcal{H}(\sigma_{(j)},\tau)$ defined by the $\pi(\sigma_{(j)},\lambda_{(j)})$'s. We refer to [1,31,37,42] for more details about these terms.

3 Weighted unipotent orbital integral

3.1 Weighted orbital integral

Let us recall the intertwining operator

$$J_{\bar{P}|P}(\sigma,\lambda)\;:\;\mathcal{H}_{\sigma,\lambda}(P)\to\mathcal{H}_{\sigma,\lambda}(\bar{P})$$

is defined by

$$\left(J_{\bar{P}|P}(\sigma,\lambda)\phi\right)(x) := \int_{\bar{N}}\phi(x\bar{n})\,d\bar{n}$$

and satisfies

$$J_{\bar{P}|P}(\sigma,\lambda)\,\pi_{\sigma,\lambda}(P)(x) = \pi_{\sigma,\lambda}(\bar{P})(x)\,J_{\bar{P}|P}(\sigma,\lambda),$$

where the notation $(\pi_{\sigma,\lambda}(P),\mathcal{H}_{\sigma,\lambda}(P))$ denotes the principal series representation with its dependence on $P = MAN$. Now let

$$J_P(\sigma,\lambda:h) = -\operatorname{Tr}(\pi_{\sigma,\lambda}(h)\,J_{\bar{P}|P}(\sigma,\lambda)^{-1}\partial_{i\lambda}J_{\bar{P}|P}(\sigma,\lambda)),$$

where $\partial_{i\lambda}$ denotes the derivative under the identification (2.2) for a family of operators $J_{\bar{P}|P}(\sigma,\lambda)$ acting on

$$L^2(K,H_\sigma) := \{\,f:K\to H_\sigma\mid f(km)=\sigma(m)^{-1}f(k),\ ||f||\in L^2(K)\,\},$$

and the space $\mathcal{H}_{\sigma,\lambda}(P)$ is identified with $L^2(K, H_\sigma)$ by restriction to K. Now we can get the invariant part of $W_\chi(h)$ by subtracting the non-invariant part as follows,

$$I_P(h) = \int_N \int_K h(knk^{-1}) \log|\log n|\, dk\, dn - \frac{1}{2}\left(\frac{1}{2\pi}\sum_{\sigma\in\hat{M}} d(\sigma)\text{p.v.}\int_{-\infty}^{\infty} J_P(\sigma,\lambda : h)\, d\lambda + \sum_{\sigma\in\hat{M}} d(\sigma)\frac{n(\sigma)}{2}\Theta_{\sigma,0}(h)\right), \tag{3.1}$$

where the notation p.v. means the Cauchy principal value of the integral and $2n(\sigma)$ is the order of the zero of $\mu(\sigma,\lambda)$ at $\lambda = 0$.

To explain $I_P(h)$, we need to introduce some more notations. Let T_M be a Cartan subgroup in M so that $T = T_M \cdot A$ is a Cartan subgroup of G. Let Σ_M be the system of the positive roots for $(\mathfrak{m}_\mathbb{C}, \mathfrak{t}_{\mathfrak{m}\mathbb{C}})$. We choose the system Σ_A of positive roots of $(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ which do not vanish on $\mathfrak{a}_\mathbb{C}$ so that Σ_A is compatible with Σ_M. Then the union of Σ_M with Σ_A gives the system of positive roots for $(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$, which is denoted by Σ_G. Let $H_\alpha \in \mathfrak{t}_\mathbb{C}$ be the co-root corresponding to $\alpha \in \pm\Sigma_G$, that is, $\alpha(H_\alpha) = 2, \alpha'(H_\alpha) \in \mathbb{Z}$ for all $\alpha, \alpha' \in \pm\Sigma_G$, and let

$$\Pi = \prod_{\alpha\in\Sigma_M} H_\alpha, \tag{3.2}$$

which is an element of the symmetric algebra $S(\mathfrak{t}_{\mathfrak{m}\mathbb{C}})$. We denote the simple reflection corresponding to α by s_α for $\alpha \in \Sigma_G$. By Corollary on p. 96 of [16] (taking $\lambda_P = \frac{\beta}{2}$ with $\beta(H_\beta) = 2$), under our normalization we have

Proposition 3.1 *For $h \in \mathcal{C}^2(G) - \mathcal{C}_0^2(G)$, where $\mathcal{C}_0^2(G)$ is the space of the cusp form on G,*

$$I_P(h) = \frac{1}{2}\cdot\frac{1}{2\pi}\sum_{\sigma\in\hat{M}}\int_{-\infty}^{\infty} \Omega(\sigma,-\lambda)\Theta_{\sigma,\lambda}(h)\, d\lambda, \tag{3.3}$$

where

$$\Omega(\sigma,\lambda) = 2d(\sigma)\psi(1) - \frac{1}{2}\sum_{\alpha\in\Sigma_A}\beta(H_\alpha)\frac{\Pi(s_\alpha\lambda_\sigma)}{\Pi(\rho_M)}\left(\psi(1+\lambda_\sigma(H_\alpha)) + \psi(1-\lambda_\sigma(H_\alpha))\right). \tag{3.4}$$

Here ψ is the digamma function and $\lambda_\sigma - \rho_M$ is the highest weight of $(\sigma, i\lambda) \in \widehat{M}\times i\mathfrak{a}$, where ρ_M denotes the half sum of the positive roots of Σ_M.

3.2 Computation for σ_k

To express $W_\chi(h)$ in terms of the elements in $\widehat{G}$, we use (3.1) and (3.4).

First, let us investigate the last term on the right side of (3.1). From (2.7), we have

$$\begin{aligned} &n(\sigma_k) = 1 \quad (0 \le k \le n-1), \quad n(\sigma_n^\pm) = 0 \quad \text{if } d = 2n+1,\\ &n(\sigma_k) = 1 \quad (0 \le k \le n-1) \qquad\qquad\qquad\quad \text{if } d = 2n. \end{aligned}$$

Next we consider the term given by $J_P(\sigma, \lambda : h)$ in (3.1). For a fixed irreducible representation τ, it is known that the Harish-Chandra C-function $C_\tau(\sigma, i\lambda)$ satisfies

$$T_\tau J_{\bar{P}|P}(\sigma,\lambda)^{-1}\partial_{i\lambda} J_{\bar{P}|P}(\sigma,\lambda) \;=\; C_\tau(\sigma, i\lambda)^{-1}\partial_{i\lambda} C_\tau(\sigma, i\lambda) T_\tau, \tag{3.5}$$

where T_τ is the projection to τ-isotypic component of $\mathcal{H}_{\sigma,\lambda}(P)$. Hence, if h is τ-type and $[\tau|_M : \sigma] \neq 0$, we have

$$J_P(\sigma, \lambda : h) \;=\; -\Theta_{\sigma,\lambda}(h)\, C_\tau(\sigma, i\lambda)^{-1}\partial_{i\lambda} C_\tau(\sigma, i\lambda). \tag{3.6}$$

By Theorem 8.2 in [7], we can derive the following equalities:

(1) When $d = 2n+1$,

$$\begin{aligned} \partial_{i\lambda}\log C_{\tau_k}(\sigma_k, i\lambda) &= \frac{1}{i\lambda + n - k} - \left(\frac{1}{i\lambda} + \cdots + \frac{1}{i\lambda + n}\right),\\ \partial_{i\lambda}\log C_{\tau_k}(\sigma_{k-1}, i\lambda) &= \frac{1}{i\lambda - n + k - 1} - \left(\frac{1}{i\lambda} + \cdots + \frac{1}{i\lambda + n}\right), \end{aligned} \tag{3.7}$$

where σ_n means $\sigma_n^\pm$.

(2) When $d = 2n$,

$$\begin{aligned} \partial_{i\lambda}\log C_{\tau_k}(\sigma_k, i\lambda) &= \frac{1}{i\lambda + n - k - \frac{1}{2}} + \left(\psi(i\lambda) - \psi\left(i\lambda + n + \frac{1}{2}\right)\right)\\ &\quad +2\log 2,\\ \partial_{i\lambda}\log C_{\tau_k}(\sigma_{k-1}, i\lambda) &= \frac{1}{i\lambda - n + k - \frac{1}{2}} + \left(\psi(i\lambda) - \psi\left(i\lambda + n + \frac{1}{2}\right)\right) + 2\log 2, \end{aligned} \tag{3.8}$$

where τ_n means $\tau_n^\pm$.

Now the remaining main task to compute $W_\chi(h)$ is to obtain an explicit form of $\Omega(\sigma_k, \lambda)$ which express $I_P(h)$ in terms of the principal series.

Theorem 3.2 *For the representations* σ_k *of* $\mathrm{SO}(d-1)$ *for* $0 \le k \le [\frac{d-1}{2}]$, *we have*

$$\Omega(\sigma_k, \lambda) = -\frac{d(\sigma_k)}{2}\,(\psi(i\lambda-(d-1)/2)+\psi(-i\lambda-(d-1)/2)+\psi(i\lambda+1)$$
$$+\,\psi(-i\lambda+1)) + \frac{(-1)^k\left(\frac{d-1}{2}-k\right)}{\lambda^2+\left(\frac{d-1}{2}-k\right)^2}$$
$$\times\left(\sum_{j=0}^{k-1}(-1)^j d(\sigma_j) + \sum_{j=k+1}^{d-1}(-1)^{j+1}d(\sigma_j)\right) - P_k(\lambda),$$

where σ_n *denotes* $\sigma_n^{\pm}$ *if* $d=2n+1$ *and* $P_k(\lambda)$ *is an even polynomial of degree* $2n-4$ *for* $d=2n+1\ge 5$ *or* $d=2n\ge 4$ *and a constant for* $d=3,2$.

The proof of this theorem will be given in the next subsection.

For τ_k, σ_ℓ with $[\tau_k|_M : \sigma_\ell] \ne 0$, we put

$$\Phi(\tau_k, \sigma_\ell, \lambda) := -d(\sigma_\ell)\frac{1}{2}\left(\partial_{i\lambda}\log C_{\tau_k}(\sigma_\ell, i\lambda) - \partial_{i\lambda}\log C_{\tau_k}(\sigma_\ell, -i\lambda)\right).$$

Then by the equalities (3.7), (3.8) and Theorem 3.2, we have

Corollary 3.3 *The following equalities hold,*

$$\Omega(\sigma_k,\lambda)+\Phi(\tau_k,\sigma_k,\lambda)$$
$$= -d(\sigma_k)\,(\psi(i\lambda+1)+\psi(-i\lambda+1)) + \frac{(-1)^k(\frac{d-1}{2}-k)}{\lambda^2+(\frac{d-1}{2}-k)^2}$$
$$\times\left(\sum_{j=0}^{k-1}(-1)^j d(\sigma_j) + \sum_{j=k}^{d-1}(-1)^{j+1}d(\sigma_j)\right) - \widetilde{P}_k(\lambda),$$
$$\Omega(\sigma_k,\lambda)+\Phi(\tau_{k+1},\sigma_k,\lambda)$$
$$= -d(\sigma_k)\,(\psi(i\lambda+1)+\psi(-i\lambda+1)) + \frac{(-1)^k(\frac{d-1}{2}-k)}{\lambda^2+(\frac{d-1}{2}-k)^2}$$
$$\times\left(\sum_{j=0}^{k}(-1)^j d(\sigma_j) + \sum_{j=k+1}^{d-1}(-1)^{j+1}d(\sigma_j)\right) - \widetilde{P}_k(\lambda),$$

where $\widetilde{P}_k(\lambda) = P_k(\lambda)$ *if* $d=2n+1$, $\widetilde{P}_k(\lambda) = P_k(\lambda)+2\log 2$ *if* $d=2n$.

Remark 3.4 For the case of τ_0, the first equality in Corollary 3.3 coincides with the formula in [11]. First, the rational part $\frac{(-1)^k(\frac{d-1}{2}-k)}{\lambda^2+(\frac{d-1}{2}-k)^2}$ vanishes in this case since $\sum_{j=0}^{d-1}(-1)^{j+1}d(\sigma_j)=0$. Recalling that $\Theta_{\sigma_0,\lambda}(h)$ is even function of λ and the factor $\frac{1}{4\pi}$ in front of integral in (3.3), we can see that the formula for $\Omega(\sigma_0,\lambda)+\Phi(\tau_0,\sigma_0,\lambda)$ coincides with (1.32) or (1.47) of [11], where the polynomial part also reduces to a constant.

3.3 Proof of Theorem 3.2

Here we present the detail of the proof only for the even dimensional case. The odd dimensional case can be dealt with in a similar way and its detail is given in [31]. For convenience of the computation, we let $n = \frac{d}{2} - 1$ so that

$$d = 2(n+1)$$

throughout this subsection. The case of $n = 0$ can be computed as in the cases $n \geq 1$. Hence we assume that $n \geq 1$ in the following proof.

With respect to the inner product on $\mathfrak{t}_{\mathbb{C}}^*$ induced from $\langle \cdot, \cdot \rangle$ in (2.1), we choose an orthonormal basis $\{e_i\}$ of $\mathfrak{t}_{\mathbb{C}}^*$ such that $e_1 \in \mathfrak{a}_{\mathbb{C}}^*$. Then we have

$$\begin{aligned}\Sigma_G &= \{e_i \ (1 \leq i \leq n+1), \quad e_i - e_j \ (1 \leq i < j \leq n+1), \\ &\qquad e_i + e_j \ (1 \leq i < j \leq n+1)\}, \\ \Sigma_A &= \{e_1, \quad e_1 - e_j \ (1 < j \leq n+1), \quad e_1 + e_j \ (1 < j \leq n+1)\}.\end{aligned}$$

Let us write λ_{σ_k} in terms of $\{e_i\}$. The highest weights μ_k of the representations σ_k of $M = \mathrm{SO}(2n+1) \subset K = \mathrm{SO}(2(n+1))$ are given by

$$\mu_k = e_2 + e_3 + \cdots + e_{k+1} \quad (0 \leq k \leq n).$$

Recalling

$$\rho_M = \left(n - \frac{1}{2}\right) e_2 + \left(n - \frac{3}{2}\right) e_3 + \cdots + \frac{1}{2} e_{n+1},$$

we have

$$\begin{aligned}\lambda_{\sigma_k} &= i\lambda e_1 + \mu_k + \rho_M \\ &= i\lambda e_1 + \left(n + \frac{1}{2}\right) e_2 + \left(n - \frac{1}{2}\right) e_3 + \cdots + \left(n - k + \frac{3}{2}\right) e_{k+1} \\ &\quad + \left(n - k - \frac{1}{2}\right) e_{k+2} \cdots + \frac{1}{2} e_{n+1}.\end{aligned}$$

First we consider $\Pi(s_\alpha \lambda_\sigma)$ for $\alpha \in \Sigma_A$, which are given by $e_1, e_1 - e_\ell, e_1 + e_\ell$ for $2 \leq \ell \leq n+1$. Then we have

$$\begin{aligned}&s_{e_1}(i\lambda e_1 + \mu_k + \rho_M) \\ &= -i\lambda e_1 + \left(n + \frac{1}{2}\right) e_2 + \left(n - \frac{1}{2}\right) e_3 + \cdots + \left(n - k + \frac{3}{2}\right) e_{k+1} \\ &\quad + \left(n - k - \frac{1}{2}\right) e_{k+2} \cdots + \frac{1}{2} e_{n+1}.\end{aligned}$$

$$s_{(e_1-e_\ell)}(i\lambda e_1+\mu_k+\rho_M)$$
$$=\begin{cases} i\lambda e_\ell+\left(n+\frac{1}{2}\right)e_2+\cdots+\left(n-\ell+\frac{5}{2}\right)e_1+\cdots+\left(n-k+\frac{3}{2}\right)e_{k+1} \\ \quad +\left(n-k-\frac{1}{2}\right)e_{k+2}+\cdots+\frac{1}{2}e_{n+1} \quad \text{if} \quad 2\le \ell\le k+1, \\ i\lambda e_\ell+\left(n+\frac{1}{2}\right)e_2+\cdots+\left(n-k+\frac{3}{2}\right)e_{k+1}+\left(n-k-\frac{1}{2}\right)e_{k+2} \\ \quad +\cdots+\left(n-\ell+\frac{3}{2}\right)e_1+\cdots+\frac{1}{2}e_{n+1} \quad \text{if} \quad k+2\le\ell\le n+1, \end{cases}$$
$$s_{(e_1+e_\ell)}(i\lambda e_1+\mu_k+\rho_M)$$
$$=\begin{cases} -i\lambda e_\ell+\left(n+\frac{1}{2}\right)e_2+\cdots-\left(n-\ell+\frac{5}{2}\right)e_1+\cdots+\left(n-k+\frac{3}{2}\right)e_{k+1} \\ \quad +\left(n-k-\frac{1}{2}\right)e_{k+2}+\cdots+\frac{1}{2}e_{n+1} \quad \text{if} \quad 2\le \ell\le k+1, \\ -i\lambda e_\ell+\left(n+\frac{1}{2}\right)e_2+\cdots+\left(n-k+\frac{3}{2}\right)e_{k+1}+\left(n-k-\frac{1}{2}\right)e_{k+2} \\ \quad +\cdots-\left(n-\ell+\frac{3}{2}\right)e_1+\cdots+\frac{1}{2}e_{n+1} \quad \text{if} \quad k+2\le\ell\le n+1. \end{cases}$$

Recall that Σ_M consists of e_i for $2\le i\le n+1$, $e_i\pm e_j$ for $2\le i<j\le n+1$ and the co-root H_α of α satisfies $\alpha(H_\alpha)=2$. By the Weyl's dimension formula, for $\alpha=e_1$,

$$\Pi\left(s_{e_1}(i\lambda e_1+\mu_k+\rho_M)\right)=d(\sigma_k)\Pi(\rho_M).$$

For the other cases, it is a polynomial of λ as follows:

$$\Pi(s_{(e_1\pm e_\ell)}(i\lambda e_1+\mu_k+\rho_M))$$
$$=C^k_{\ell-1}(\mp i\lambda)\left(\lambda^2+\left(n+\frac{1}{2}\right)^2\right)\cdots\left(\lambda^2+\left(n-\ell+\frac{7}{2}\right)^2\right)\cdot\left(-\lambda^2-\left(n-\ell+\frac{3}{2}\right)^2\right)\cdots$$
$$\left(-\lambda^2-\left(n-k+\frac{3}{2}\right)^2\right)\cdot\left(-\lambda^2-\left(n-k-\frac{1}{2}\right)^2\right)\cdots\left(-\lambda^2-\left(\frac{1}{2}\right)^2\right)$$
$$\text{if } 2\le\ell\le k+1,$$
$$\Pi(s_{(e_1\pm e_\ell)}(i\lambda e_1+\mu_k+\rho_M))$$
$$=C^k_{\ell}(\mp i\lambda)\left(\lambda^2+\left(n+\frac{1}{2}\right)^2\right)\cdots$$
$$\left(\lambda^2+\left(n-k+\frac{3}{2}\right)^2\right)\cdot\left(\lambda^2+\left(n-k-\frac{1}{2}\right)^2\right)\cdots\left(\lambda^2+\left(n-\ell+\frac{5}{2}\right)^2\right)$$
$$\cdot\left(-\lambda^2-\left(n-\ell+\frac{1}{2}\right)^2\right)\cdots\left(-\lambda^2-\left(\frac{1}{2}\right)^2\right) \quad \text{if } k+2\le\ell\le n+1,$$

where

$$C^k_\ell=2^n\prod_{\substack{0\le a<b\le n\\ a,b\notin\{n-k,n-\ell\}}}\left(\left(b+\frac{1}{2}\right)^2-\left(a+\frac{1}{2}\right)^2\right)\cdot\prod_{\substack{0\le c\le n\\ c\notin\{n-k,n-\ell\}}}\left(c+\frac{1}{2}\right)$$

for $0 \le k \le n$, $2 \le \ell \le n+1$. By the above computation, we can put

$$P^n_{k,\ell}(\lambda) := \Pi(s_{e_1 \pm e_\ell}(i\lambda e_1 + \mu_k + \rho_M))$$

which is a degree $2n-3$ odd polynomial of λ.

Second we compute the part $(\psi(1+\lambda_\sigma(H_\alpha)) + \psi(1-\lambda_\sigma(H_\alpha))$ for $\alpha \in \Sigma_A$. For this,

$$\begin{aligned}
&(i\lambda e_1 + \mu_k + \rho_M)(H_\alpha)\\
&\quad = \begin{cases}
2i\lambda & \text{if } \alpha = 2e_1 \\
i\lambda - \left(n - \ell + \frac{5}{2}\right) & \text{if } \alpha = e_1 - e_\ell, \quad 2 \le \ell \le k+1 \\
i\lambda - \left(n - \ell + \frac{3}{2}\right) & \text{if } \alpha = e_1 - e_\ell, \quad k+2 \le \ell \le n+1 \\
i\lambda + \left(n - \ell + \frac{5}{2}\right) & \text{if } \alpha = e_1 + e_\ell, \quad 2 \le \ell \le k+1 \\
i\lambda + \left(n - \ell + \frac{3}{2}\right) & \text{if } \alpha = e_1 + e_\ell, \quad k+2 \le \ell \le n+1.
\end{cases}
\end{aligned}$$

From this, we can see that $(\psi(1+(i\lambda e_1+\mu_k+\rho_M)(H_\alpha)), \psi(1-(i\lambda e_1+\mu_k+\rho_M)(H_\alpha))$ is given by

$$\begin{array}{lll}
\psi(2i\lambda+1),\ \psi(-2i\lambda+1) & \text{for } \alpha = 2e_1, & \\
\psi\left(i\lambda - n + \ell - \frac{3}{2}\right),\ \psi\left(-i\lambda + n - \ell + \frac{7}{2}\right) & \text{for } \alpha = e_1 - e_\ell, & 2 \le \ell \le k+1 \\
\psi\left(i\lambda - n + \ell - \frac{1}{2}\right),\ \psi\left(-i\lambda + n - \ell + \frac{5}{2}\right) & \text{for } \alpha = e_1 - e_\ell, & k+2 \le \ell \le n+1 \\
\psi\left(i\lambda + n - \ell + \frac{7}{2}\right),\ \psi\left(-i\lambda - n + \ell - \frac{3}{2}\right) & \text{for } \alpha = e_1 + e_\ell, & 2 \le \ell \le k+1 \\
\psi\left(i\lambda + n - \ell + \frac{5}{2}\right),\ \psi\left(-i\lambda - n + \ell - \frac{1}{2}\right) & \text{for } \alpha = e_1 + e_\ell, & k+2 \le \ell \le n+1.
\end{array}$$

For the sum over $\alpha \in \Sigma_A$ in (3.4), we first consider the term with $\alpha = e_1$. By the results obtained above,

$$\begin{aligned}
&\frac{1}{2}\beta(H_{e_1})\frac{\Pi(s_{e_1}\lambda_\sigma)}{\Pi(\rho_M)}\left(\psi(1+\lambda_\sigma(H_{e_1})) + \psi(1-\lambda_\sigma(H_{e_1}))\right)\\
&\quad = d(\sigma_k)\left(\psi(2i\lambda+1) + \psi(-2i\lambda+1)\right)\\
&\quad = \frac{d(\sigma_k)}{2}\left(\psi\left(i\lambda+\frac{1}{2}\right) + \psi\left(-i\lambda+\frac{1}{2}\right) + \psi(i\lambda+1) + \psi(-i\lambda+1) + 4\log 2\right)\\
&\quad = \frac{d(\sigma_k)}{2}\left(\psi\left(i\lambda-n-\frac{1}{2}\right) + \psi\left(-i\lambda-n-\frac{1}{2}\right) + \psi(i\lambda+1) + \psi(-i\lambda+1)\right.\\
&\qquad \left. + \frac{-2\cdot\frac{1}{2}}{\lambda^2+\left(\frac{1}{2}\right)^2} + \cdots + \frac{-2\left(n+\frac{1}{2}\right)}{\lambda^2+\left(n+\frac{1}{2}\right)^2} + 4\log 2\right)
\end{aligned} \tag{3.9}$$

by the properties of the digamma function $\psi(z)$. Now we take a sum over e_1+e_ℓ, e_1-e_ℓ in (3.4). For $2 \le \ell \le k+1$,

$$\begin{aligned}
&\frac{1}{2}\sum_{\alpha=e_1\pm e_\ell}\beta(H_\alpha)\frac{\Pi(s_\alpha\lambda_\sigma)}{\Pi(\rho_M)}\left(\psi(1+\lambda_\sigma(H_\alpha))+\psi(1-\lambda_\sigma(H_\alpha))\right)\\
&=\frac{1}{2}\frac{\Pi(s_{e_1-e_\ell}\lambda_\sigma)}{\Pi(\rho_M)}\left(\psi\left(i\lambda-n+\ell-\frac{3}{2}\right)+\psi\left(-i\lambda+n-\ell+\frac{7}{2}\right)\right.\\
&\left.-\psi\left(i\lambda+n-\ell+\frac{7}{2}\right)-\psi\left(-i\lambda-n+\ell-\frac{3}{2}\right)\right)\\
&=\frac{1}{2}\frac{\Pi(s_{e_1-e_\ell}\lambda_\sigma)}{\Pi(\rho_M)}\left(\frac{4i\lambda}{\lambda^2+\left(\frac{1}{2}\right)^2}+\cdots+\frac{4i\lambda}{\lambda^2+\left(n-\ell+\frac{3}{2}\right)^2}+\frac{2i\lambda}{\lambda^2+\left(n-\ell+\frac{5}{2}\right)^2}\right),
\end{aligned} \tag{3.10}$$

and similarly for $k+2\leq\ell\leq n+1$,

$$\begin{aligned}
&\frac{1}{2}\sum_{\alpha=e_1\pm e_\ell}\beta(H_\alpha)\frac{\Pi(s_\alpha\lambda_\sigma)}{\Pi(\rho_M)}\left(\psi(1+\lambda_\sigma(H_\alpha))+\psi(1-\lambda_\sigma(H_\alpha))\right)\\
&=\frac{1}{2}\frac{\Pi(s_{e_1-e_\ell}\lambda_\sigma)}{\Pi(\rho_M)}\left(\frac{4i\lambda}{\lambda^2+\left(\frac{1}{2}\right)^2}+\cdots+\frac{4i\lambda}{\lambda^2+\left(n-\ell+\frac{1}{2}\right)^2}+\frac{2i\lambda}{\lambda^2+\left(n-\ell+\frac{3}{2}\right)^2}\right).
\end{aligned} \tag{3.11}$$

From the expression of $\Pi(s_{e_1-e_\ell}\lambda_\sigma)$, we can see that the term in (3.10), (3.11) consists of a polynomial of degree $2n-2$ if $d=2(n+1)\geq 4$ and some rational functions whose denominators are $\lambda^2+(n-k+\frac{1}{2})^2$, $\lambda^2+(n-\ell+\frac{5}{2})^2$ when $2\leq\ell\leq k+1$ and $\lambda^2+(n-\ell+\frac{3}{2})^2$ when $k+2\leq\ell\leq n+1$. The numerators of these rational functions are given by

$$\begin{aligned}
2i\lambda\frac{P^n_{k,\ell}(\lambda)}{\Pi(\rho_M)}\Big|_{\lambda=i\left(n-k+\frac{1}{2}\right)} &= (-1)^{k-\ell-1}2\left(n-k+\frac{1}{2}\right)d(\sigma_\ell) \qquad \text{for } 2\leq\ell\leq k+1,\\
i\lambda\frac{P^n_{k,\ell}(\lambda)}{\Pi(\rho_M)}\Big|_{\lambda=i(n-\ell+\frac{5}{2})} &= \left(n-\ell+\frac{5}{2}\right)d(\sigma_k) \qquad \text{for } 2\leq\ell\leq k+1,\\
i\lambda\frac{P^n_{k,\ell}(\lambda)}{\Pi(\rho_M)}\Big|_{\lambda=i\left(n-\ell+\frac{3}{2}\right)} &= (n-\ell+\frac{3}{2})d(\sigma_k) \qquad \text{for } k+2\leq\ell\leq n+1,
\end{aligned}$$

so that the sum of these rational functions over $2\leq\ell\leq n+1$ is

$$\begin{aligned}
&(-1)^k 2\sum_{1\leq\ell\leq k}(-1)^\ell d(\sigma_{\ell-1})\,\frac{\left(n-k+\frac{1}{2}\right)}{\lambda^2+\left(n-k+\frac{1}{2}\right)^2}\\
&+d(\sigma_k)\sum_{\substack{2\leq\ell\leq n+1\\ \ell\neq k+1}}\frac{\left(n-\ell+\frac{3}{2}\right)}{\lambda^2+\left(n-\ell+\frac{3}{2}\right)^2}.
\end{aligned} \tag{3.12}$$

Finally taking the terms in (3.9) and (3.12) with a polynomial denoted by $P_k(\lambda)$, we obtain

$$\Omega(\sigma_k,\lambda) = -\frac{d(\sigma_k)}{2}\left(\psi\left(i\lambda-n-\frac{1}{2}\right)+\psi\left(-i\lambda-n-\frac{1}{2}\right)+\psi(i\lambda+1)\right.$$
$$\left.+\psi(-i\lambda+1)\right)+\frac{(-1)^k\left(n-k+\frac{1}{2}\right)}{\lambda^2+\left(n-k+\frac{1}{2}\right)^2}$$
$$\times\left(\sum_{j=0}^{k-1}(-1)^j d(\sigma_j)+\sum_{j=k+1}^{2n+1}(-1)^{j+1}d(\sigma_j)\right)-P_k(\lambda).$$

4 Selberg zeta functions

4.1 Paley–Wiener theorem

We put $PW_R(\mathbb{C})$ to be the set whose elements are entire functions h on $\mathbb{C}$ such that for any $n\in\mathbb{N}$, there exists $C_n>0$ with

$$|h(\lambda)|\le C_n(1+|\lambda|)^{-n}e^{R|\operatorname{Im}(\lambda)|}\quad\text{over }\ \lambda\in\mathbb{C},$$

and $PW_R^e(\mathbb{C})$ to be a subset which consists of even functions in $PW_R(\mathbb{C})$. For $\tau=\tau_k$, we also put $C_R^\infty(G,K,\tau)$ to be the set of τ-radial function $f:G\to\operatorname{End}(V_\tau)$ such that

$$f(k_1gk_2)=\tau(k_2)^{-1}f(g)\tau(k_1)^{-1}$$

for $k_1,k_2\in K$, $g\in G$ and its support is in the closed ball $B(eK,R)$ in G/K of hyperbolic radius R. If $f\in C_R^\infty(G,K,\tau)$, $\operatorname{tr}(f)$ is a scalar function on G, where tr denotes the trace over V_τ. By the Paley-Wiener theorem stated in Theorem 3 of [33], whose full proof can be found in Theorem 6.5, 6.12, 6.14 of [34], there exists a τ_k-radial function $f^k\in C_R^\infty(G,K,\tau_k)$ such that

$$\Theta_{\sigma_\ell,\lambda}\left(\operatorname{tr}(f^k)\right)=H^\ell(\lambda),\quad \ell=k,k-1$$

for $k\neq n$ with $d=2n$ or $d=2n+1$ if $H^k(\lambda),H^{k-1}(\lambda)\in PW_R^e(\mathbb{C})$ with

$$H^k\left(\pm i\,d_k\right)=\ H^{k-1}\left(\pm i\,d_{k-1}\right),\quad \text{where } d_\ell=(d-1)/2-\ell, \tag{4.1}$$

for $k=n$ and $d=2n+1$ if $H^{n,\pm}(\lambda)\in PW_R(\mathbb{C})$, $H^{n-1}(\lambda)\in PW_R^e(\mathbb{C})$ with

$$H^{n,\pm}(\lambda)=H^{n,\mp}(-\lambda),\quad H^{n,\pm}(0)=H^{n-1}(\pm i), \tag{4.2}$$

for the case of $k=n$ and $d=2n$ if $H^n(\lambda)\in PW_R^e(\mathbb{C})$. Note that there is no matching condition for this case since only σ_n satisfies $[\tau_n^\pm|_M:\sigma_n]\neq 0$.

To apply the Paley-Wiener theorem explained above, for an even function $g_\ell(x) \in C_c^\infty(\mathbb{R})$ whose support lies in $[-R, R] \subset \mathbb{R}$, we define

$$H(g_\ell, \lambda) := \hat{g}_\ell(\lambda) = \int_{-\infty}^{\infty} g_\ell(x) e^{i\lambda x}\, dx,$$

which lies in $PW_R^e(\mathbb{C})$. When $k \neq n$ with $d = 2n$ or $d = 2n+1$, we put $H^k(\lambda) = H(g_k, \lambda)$ and $H^{k-1}(\lambda) = h_k H(g_{k-1}, \lambda)$, where $h_k = H(g_k, \pm i\, d_k) H(g_{k-1}, \pm i\, d_{k-1})^{-1}$ for the matching condition in (4.1). When $k = n$ and $d = 2n+1$, we put $H^{n,\pm}(\lambda) = H(g_n, \lambda)$ and $H^{n-1}(\lambda) = h_n H(g_{n-1}, \lambda)$, where $h_n = H^{n,\pm}(0)\, H^{n-1}(\pm i)^{-1}$ for the matching condition (4.2). When $k = n$ and $d = 2n$, we put $H^n(\lambda) = H(g_n, \lambda)$. Now by the Paley-Wiener theorem, for $k \neq n$ with $d = 2n$ or $d = 2n+1$, we have τ_k-radial function $f^k \in C_R^\infty(G, K, \tau_k)$ such that

$$\begin{aligned} \Theta_{\sigma_k,\lambda}\left(\mathrm{tr}(f^k)\right) &= H^k(\lambda) = H(g_k, \lambda), \\ \Theta_{\sigma_{k-1},\lambda}\left(\mathrm{tr}(f^k)\right) &= H^{k-1}(\lambda) = h_k H(g_{k-1}, \lambda). \end{aligned} \tag{4.3}$$

For other cases, we also have the similar results.

Combining the results proved in the previous sections, the Selberg trace formula applied to the test function $\mathrm{tr}(f^k) \in \mathcal{C}^p(G)$ $(0 < p < 1)$ with the property (4.3) has the following form:

$$\begin{aligned} &\sum_{[\tau_k|_M:\sigma_\ell]\neq 0} \Bigg(\sum_j m_j(\ell)\, H^\ell(\lambda_j(\ell)) + \frac{d(\sigma_\ell)}{4} H^\ell(0)\, \mathrm{tr}\left(C_\chi^k(\sigma_\ell, 0) - n(\sigma_\ell)\mathrm{Id}\right) \\ &\quad - \frac{d(\sigma_\ell)}{4\pi\sqrt{-1}} \int_{-\infty}^{\infty} H^\ell(\lambda)\, \mathrm{tr}\left(C_\chi^k(\sigma_\ell, -i\lambda) \frac{d}{d\lambda} C_\chi^k(\sigma_\ell, i\lambda)\right) d\lambda \Bigg) \\ &= \sum_{[\tau_k|_M:\sigma_\ell]\neq 0} \Bigg(\dim V_\chi\, \mathrm{Vol}(\Gamma\backslash G) \cdot \frac{1}{4\pi} \int_{-\infty}^{\infty} H^\ell(\lambda)\, \mu(\sigma_\ell, \lambda)\, d\lambda \\ &\quad + \sum_{\{\gamma\}_\Gamma \in \Gamma_{\mathrm{hyp}}} \mathrm{tr}\,\chi(\gamma)\, l(C_\gamma)\, j(\gamma)^{-1}\, D(\gamma)^{-1}\, \overline{\mathrm{tr}\,\sigma_\ell(m_\gamma)}\, \frac{1}{2\pi} \int_{-\infty}^{\infty} H^\ell(\lambda) e^{-il(C_\gamma)\lambda}\, d\lambda \\ &\quad + c_\chi \int_{-\infty}^{\infty} H^\ell(\lambda)\, d\lambda + \frac{d_c(\chi)}{4\pi} \int_{-\infty}^{\infty} H^\ell(\lambda)\, \Omega(\sigma_\ell, -\lambda)\, d\lambda \\ &\quad - \frac{d_c(\chi)\, d(\sigma_\ell)}{4\pi\sqrt{-1}} \mathrm{p.v.} \int_{-\infty}^{\infty} H^\ell(\lambda)\, C_{\tau_k}(\sigma_\ell, i\lambda)^{-1} \frac{d}{d\lambda} C_{\tau_k}(\sigma_\ell, i\lambda)\, d\lambda \Bigg). \end{aligned} \tag{4.4}$$

Here $\{\lambda_j(\ell)^2+(\frac{d-1}{2}-\ell)^2 \mid [\tau_k|_M : \sigma_\ell] \neq 0\}$ is the set of the discrete eigenvalues with multiplicities $m_j(\ell)$'s of the Hodge Laplacian Δ_k acting on the L^2-space of k-forms twisted by χ, c_χ is a certain constant depending on χ.

Remark 4.1 Concerning (4.4), some remarks are in need for the case when $d = 2n+1$ and $k = n$. In this case the hyperbolic terms consist of a sum of the above expression for $\sigma = \sigma_n^{\pm}$. Since σ_n^+, σ_n^- are un-ramified, the scattering matrix $C_\chi^n(\sigma_n, s)$ has the size of $2d_c(\chi) \times 2d_c(\chi)$. The last three terms on the right hand side have to be doubled since we have the same values for $\sigma = \sigma_n^{\pm}$. The forthcoming equalities for this case should be understood as we remarked now.

Proposition 4.2 *For $H^k(\lambda) \in PW_R^e(\mathbb{C})$, the following equality holds*:

$$\sum_{\{\gamma\}_\Gamma \in \Gamma_{\mathrm{hyp}}} \operatorname{tr} \chi(\gamma)\, l(C_\gamma)\, j(\gamma)^{-1}\, D(\gamma)^{-1}\, \overline{\operatorname{tr} \sigma_k(m_\gamma)}\, \frac{1}{2\pi} \int_{-\infty}^{\infty} H^k(\lambda) e^{-il(C_\gamma)\lambda}\, d\lambda$$
$$= \sum_j m_j(k)\, H^k(\lambda_j(k)) + \frac{d(\sigma_k)}{4} H^k(0) \operatorname{tr}\left(C_\chi^k(\sigma_k, 0) - n(\sigma_k)\mathrm{Id}\right)$$
$$-\frac{d(\sigma_k)}{4\pi\sqrt{-1}} \int_{-\infty}^{\infty} H^k(\lambda) \operatorname{tr}\left(C_\chi^k(\sigma_k, -i\lambda) \frac{d}{d\lambda} C_\chi^k(\sigma_k, i\lambda)\right) d\lambda$$
$$-\dim V_\chi \operatorname{Vol}(\Gamma\backslash G) \cdot \frac{1}{4\pi} \int_{-\infty}^{\infty} H^k(\lambda)\, \mu(\sigma_k, \lambda)\, d\lambda$$
$$-c_\chi \int_{-\infty}^{\infty} H^k(\lambda)\, d\lambda - \frac{d_c(\chi)}{4\pi} \int_{-\infty}^{\infty} H^k(\lambda)\, \Omega(\sigma_k, -\lambda)\, d\lambda$$
$$+\frac{d_c(\chi)\, d(\sigma_k)}{4\pi\sqrt{-1}} \mathrm{p.v.} \int_{-\infty}^{\infty} H^k(\lambda)\, C_{\tau_k}(\sigma_k, i\lambda)^{-1} \frac{d}{d\lambda} C_{\tau_k}(\sigma_k, i\lambda)\, d\lambda. \tag{4.5}$$

When $d = 2n+1$ and $k = n$, this equality should be understood as in Remark 4.1.

Proof For $\ell = k, k-1$, we put

$$I_\ell^k(g_\ell) := \sum_j m_j(\ell)\, H^\ell(\lambda_j(\ell)) - \frac{d_c(\chi)\, d(\sigma_\ell)}{4} H^\ell(0)\, n(\sigma_\ell)$$
$$-\dim V_\chi \operatorname{Vol}(\Gamma\backslash G) \cdot \frac{1}{4\pi} \int_{-\infty}^{\infty} H^\ell(\lambda)\, \mu(\sigma_\ell, \lambda)\, d\lambda$$

$$
\begin{aligned}
&-\sum_{\{\gamma\}_\Gamma \in \Gamma_{\mathrm{hyp}}} \operatorname{tr}\chi(\gamma)\, l(C_\gamma)\, j(\gamma)^{-1}\, D(\gamma)^{-1}\, \overline{\operatorname{tr}\sigma_\ell(m_\gamma)}\, \frac{1}{2\pi} \int_{-\infty}^{\infty} H^\ell(\lambda) e^{-il(C_\gamma)\lambda}\, d\lambda \\
&-c_\chi \int_{-\infty}^{\infty} H^\ell(\lambda)\, d\lambda - \frac{d_c(\chi)}{4\pi} \int_{-\infty}^{\infty} H^\ell(\lambda)\, \Omega(\sigma_\ell, -\lambda)\, d\lambda,
\end{aligned}
$$

$$
\begin{aligned}
J_\ell^k(g_\ell) :=\ & \frac{d(\sigma_\ell)}{4\pi\sqrt{-1}} \int_{-\infty}^{\infty} H^\ell(\lambda) \operatorname{tr}\left(C_\chi^k(\sigma_\ell, -i\lambda) \frac{d}{d\lambda} C_\chi^k(\sigma_\ell, i\lambda) \right) d\lambda \\
& - \frac{d_c(\chi)\, d(\sigma_\ell)}{4\pi\sqrt{-1}} \mathrm{p.v.} \int_{-\infty}^{\infty} H^\ell(\lambda)\, C_{\tau_k}(\sigma_\ell, i\lambda)^{-1} \frac{d}{d\lambda} C_{\tau_k}(\sigma_\ell, i\lambda)\, d\lambda \\
& - \frac{d(\sigma_\ell)}{4} H^\ell(0) \operatorname{tr}\left(C_\chi^k(\sigma_\ell, 0) \right). \qquad (4.6)
\end{aligned}
$$

From (4.4), we have the equality $I_k^k(g_k) + I_{k-1}^k(g_{k-1}) = J_k^k(g_k) + J_{k-1}^k(g_{k-1})$. Let us consider

$$
\left(I_k^k - J_k^k \right)(g_k) = \left(J_{k-1}^k - I_{k-1}^k \right)(g_{k-1}), \qquad (4.7)
$$

where we regard both sides as the distributions valuated at the test functions g_k, $g_{k-1} \in C_c^\infty(\mathbb{R})$. Now we vary g_{k-1} along a family of even functions $g_{k-1,t} := \alpha_0 + t\alpha_1 \in C_c^\infty(\mathbb{R})$, $t \in (-\delta, \delta)$ with keeping g_k fixed, where α_0, α_1 are even functions in $C_c^\infty(\mathbb{R})$. Taking the derivative with respect to t of both sides of (4.7) valuated to $g_k, g_{k-1,t}$ and putting $t = 0$, we obtain

$$
\left(J_{k-1}^k - I_{k-1}^k \right)(\alpha_1 - H(\alpha_1, \pm i d_{k-1}) H(\alpha_0, \pm i d_{k-1})^{-1} \alpha_0) = 0. \qquad (4.8)
$$

Note that the even function $\alpha_1 - H(\alpha_1, \pm i d_{k-1}) H(\alpha_0, \pm i d_{k-1})^{-1} \alpha_0$ can be arbitrary in $C_c^\infty(\mathbb{R})$, hence we have $I_{k-1}^k = J_{k-1}^k$, which also implies $I_k^k = J_k^k$ in the distributional sense by (4.7). Since the Fourier transform of an even function $g_\ell(x) \in C_c^\infty(\mathbb{R})$ whose support lies in $[-R, R] \subset \mathbb{R}$ gives any element in $PW_R^e(\mathbb{C})$ by the classical Paley-Wiener theorem, the equality (4.5) holds for $H(\lambda) \in PW_R^e(\mathbb{C})$. □

We multiply the inverse of the Harish-Chandra C-function $C_{\tau_k}(\sigma_\ell, s)^{-1}$ with $C_\chi^k(\sigma_\ell, s)$ to define

$$
S_\chi^k(\sigma_\ell, s) := C_{\tau_k}(\sigma_\ell, s)^{-1} C_\chi^k(\sigma_\ell, s), \qquad (4.9)
$$

which acts on the same space as $C_\chi^k(\sigma_\ell, s)$. Let us recall that $C_{\tau_k}(\sigma_\ell, s)$ is a scalar function satisfying (3.5).

Corollary 4.3 *For $H^k(\lambda) \in PW_R^e(\lambda)$, we have*

$$\text{p.v.} \int_{-\infty}^{\infty} H^k(\lambda)\,\mathrm{tr}\left(\partial_{i\lambda}\, S_\chi^{k+1}(\sigma_k, i\lambda)\right) d\lambda = \text{p.v.} \int_{-\infty}^{\infty} H^k(\lambda)\,\mathrm{tr}\left(\partial_{i\lambda}\, S_\chi^{k}(\sigma_k, i\lambda)\right) d\lambda \tag{4.10}$$

and $\mathrm{tr}(C_\chi^{k+1}(\sigma_k, 0)) = \mathrm{tr}(C_\chi^k(\sigma_k, 0))$.

Proof Recalling $I_k^{k+1} = I_k^k$ (by the definition) and the equalities $I_k^{k+1} = J_k^{k+1}$, $I_k^k = J_k^k$ derived in Proposition 4.2,

$$\begin{aligned}&\text{p.v.} \int_{-\infty}^{\infty} H^k(\lambda)\,\mathrm{tr}\left(\partial_{i\lambda} S_\chi^{k+1}(\sigma_k, i\lambda)\right) d\lambda - \pi H^k(0)\,\mathrm{tr}\left(C_\chi^{k+1}(\sigma_k, 0)\right)\\ &\quad = \text{p.v.} \int_{-\infty}^{\infty} H^k(\lambda)\,\mathrm{tr}\left(\partial_{i\lambda} S_\chi^{k}(\sigma_k, i\lambda)\right) d\lambda - \pi H^k(0)\,\mathrm{tr}\left(C_\chi^{k}(\sigma_k, 0)\right).\end{aligned}$$

As in the proof of Proposition 4.2, we vary g_k along a linear combination of even functions α_0, α_1 in $C_c^\infty(\mathbb{R})$ with keeping $H^k(0) = \int_{-\infty}^{\infty} g_k(x)\,dx$ fixed. Then we can conclude that the claimed equalities hold separately. □

Remark 4.4 The equality (4.10) also follows from the fact the operator $S(h, \tilde{w}, \sigma_\Lambda)$ in Corollary 7.2 of [17] does not depend on K-type. Although this operator $S(h, \tilde{w}, \sigma_\Lambda)$ does not reduce to the operator $S_\chi^k(\sigma_\ell, s)$ in our specific case, these are designed for the same purpose to cancel out the dependence on K-type.

Let ε_0 be the minimum of $l(C_\gamma)$ for $\gamma \in \Gamma_{\text{hyp}}$. Let $g(x)$ be a smooth even function on $\mathbb{R}$ such that $g(x) = 1$ for $|x| \geq \varepsilon_0$ and $g(x)$ vanishes in some neighborhood of zero. For $\lambda \in \mathbb{C}$, let

$$H(\lambda) := \int_0^\infty g'(x) e^{i\lambda x}\,dx.$$

Because of the properties of $g(x)$, we see that $g'(x) \in C_c^\infty(\mathbb{R})$ and $g'(x) = 0$ if $|x| > \varepsilon_0$. Hence $H(0) = g(\varepsilon_0) = 1$. The following lemma easily follows from the definition of $H(\lambda)$.

Lemma 4.5 *$H(\lambda)$ is an entire function on $\mathbb{C}$. For any integer $n \geq 1$, we can find the positive constants C_n such that*

$$|H(\lambda)| \leq \begin{cases} C_n(1+|\lambda|)^{-n} & \mathrm{Im}\,\lambda \geq 0 \\ C_n(1+|\lambda|)^{-n}\exp(\varepsilon_0|\mathrm{Im}(\lambda)|) & \mathrm{Im}\,\lambda < 0 \end{cases}.$$

Now we choose a sequence of even functions $\eta_m \in C_c^\infty(\mathbb{R})$ which is 1 over $[-m, m]$ and has the support in $(-m-1, m+1)$. For a fixed $s \in \mathbb{C}$ with $\mathrm{Re}(s) > (d-1)$, let us consider the following even function

$$H_m(s, \lambda) := \int_{-\infty}^{\infty} \eta_m(x) g(x) e^{-(s-d_0)|x|}\, e^{i\lambda x}\, dx, \quad \text{where } d_0 = (d-1)/2.$$

Since $H_m(s, \lambda)$ lies in $PW_R^e(\mathbb{C})$ for $R > (m+1)$, the equality (4.5) holds with the test function $H^k(\lambda) = H_m(s, \lambda)$. Now, we want to let $m \to \infty$ to these equalities. By the dominated convergence theorem, the limiting equality holds and both sides take a finite value if $\lim_{m\to\infty} H_m(s, \lambda)$ exits and the limit function also rapidly decays as $|\mathrm{Re}(\lambda)| \to \infty$ in the strip $\{\lambda \in \mathbb{C} \mid -d_0 \leq \mathrm{Im}(\lambda) \leq d_0\}$. Again using the dominated convergence theorem, it is easy to see that as $m \to \infty$, $H_m(s, \lambda)$ converges to the following function,

$$\begin{aligned} H(s, \lambda) &:= \int_{-\infty}^{\infty} g(x) e^{-(s-d_0)|x|}\, e^{i\lambda x}\, dx \\ &= (s-d_0-i\lambda)^{-1} H\,(i(s-d_0)+\lambda) + (s-d_0+i\lambda)^{-1} H\,(i(s-d_0)-\lambda). \end{aligned} \tag{4.11}$$

Note that $H(s, \lambda)$ has no poles in the strip $\{\lambda \in \mathbb{C} \mid -d_0 \leq \mathrm{Im}(\lambda) \leq d_0\}$ since $\mathrm{Re}(s) > (d-1)$, and rapidly decays as $|\mathrm{Re}(\lambda)| \to \infty$ by Lemma 4.5. Hence, we finally obtain the following equality: for $\mathrm{Re}(s) > (d-1)$,

$$\begin{aligned} &\sum_{\{\gamma\}_\Gamma \in \Gamma_{\mathrm{hyp}}} \mathrm{tr}\, \chi(\gamma)\, l(C_\gamma)\, j(\gamma)^{-1}\, D(\gamma)^{-1}\, \overline{\mathrm{tr}\, \sigma_k(m_\gamma)}\, \exp\left(-(s-d_0)\, l(C_\gamma)\right) \\ &= \sum_j m_j(k)\, H(s, \lambda_j(k)) + \frac{d(\sigma_k)}{4} H(s, 0)\, \mathrm{tr}\left(C_\chi^k(\sigma_k, 0) - n(\sigma_k)\mathrm{Id}\right) \\ &\quad - \frac{d(\sigma_k)}{4\pi} \mathrm{p.v.} \int_{-\infty}^{\infty} H(s, \lambda)\, \mathrm{tr}\left(S'_\chi(\sigma_k, \lambda) S_\chi(\sigma_k, \lambda)^{-1}\right)\, d\lambda \\ &\quad - \dim V_\chi\, \mathrm{Vol}(\Gamma\backslash G) \cdot \frac{1}{4\pi} \int_{-\infty}^{\infty} H(s, \lambda)\, \mu(\sigma_k, \lambda)\, d\lambda \\ &\quad - c_\chi \int_{-\infty}^{\infty} H(s, \lambda)\, d\lambda - \frac{d_c(\chi)}{4\pi} \int_{-\infty}^{\infty} H(s, \lambda)\, \Omega(\sigma_k, -\lambda)\, d\lambda \\ &=: A_{\mathrm{dis}}(s) + A_{\mathrm{res}}(s) + A_{\mathrm{sct}}(s) + A_{\mathrm{id}}(s) + A_{\mathrm{up}}(s) + A_{\mathrm{wup}}(s). \end{aligned} \tag{4.12}$$

Here $S_\chi(\sigma_k, s)$ is $S_\chi^{k+1}(\sigma_k, s)$ or $S_\chi^k(\sigma_k, s)$, which does not depend on K-type by Corollary 4.3.

4.2 Meromorphic extension of Selberg zeta function

For $0 \le k \le [\frac{d-1}{2}]$, the Selberg zeta function is defined by

$$\begin{aligned} Z_\chi(\sigma_k, s) &:= \prod_{\{\gamma\}_\Gamma \in \mathrm{P}\Gamma_{\mathrm{hyp}}} \prod_{k=0}^{\infty} \det\Big(\mathrm{Id} - \overline{\sigma_k(m_\gamma)} \otimes \chi(\gamma) \otimes S^k(\mathrm{Ad}(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}}) e^{-sl(C_\gamma)}\Big) \\ &= \exp\left(-\sum_{\gamma \in \Gamma_{\mathrm{hyp}}} \mathrm{tr}\, \chi(\gamma) j(\gamma)^{-1} D(\gamma)^{-1} \overline{\mathrm{tr}\, \sigma_k(m_\gamma)} e^{-(s-\frac{d-1}{2})l(C_\gamma)}\right) \end{aligned} \tag{4.13}$$

for $\mathrm{Re}(s) > (d-1)$. Here $\mathrm{P}\Gamma_{\mathrm{hyp}}$ denotes the set of Γ-conjugacy classes $\{\gamma\}_\Gamma$ of the primitive hyperbolic elements γ in Γ, S^k denotes the kth symmetric power of $\mathrm{Ad}(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}}$, and $\bar{\mathfrak{n}} = \theta\mathfrak{n}$. The equality (4.13) can be proved as in Lemma 3.3 of [3]. When $d = 2n+1$, we put $Z_\chi(\sigma_n, s) = Z_\chi(\sigma_n^+, s) \cdot Z_\chi(\sigma_n^-, s)$. Then we have

$$\begin{aligned} \frac{d}{ds} \log Z_\chi(\sigma_k, s) &= \sum_{\{\gamma\}_\Gamma \in \Gamma_{\mathrm{hyp}}} \mathrm{tr}\, \chi(\gamma) l(C_\gamma) j(\gamma)^{-1} D(\gamma)^{-1} \overline{\mathrm{tr}\, \sigma_k(m_\gamma)} e^{-(s-\frac{d-1}{2})l(C_\gamma)} \\ &= H_\chi(s) \quad \text{for} \quad \mathrm{Re}(s) > (d-1), \end{aligned} \tag{4.14}$$

where $H_\chi(s)$ denotes the sum of hyperbolic terms, that is, the left hand side in (4.12). By (4.12) and (4.14), now we show that $Z_\chi(\sigma_k, s)$ has a meromorphic extension to $\mathbb{C}$.

Theorem 4.6 *For $0 \le k \le [\frac{d-1}{2}]$, the Selberg zeta function $Z_\chi(\sigma_k, s)$ defined for $\mathrm{Re}(s) > (d-1)$ has a meromorphic extension to $\mathbb{C}$ with*

(1) *zeros at*
$s = \frac{d-1}{2} \pm i\lambda_j(k)$ of order $m_j(k)$, where $\lambda_j(k)^2 + (\frac{d-1}{2} - k)^2$ is an eigenvalue of Δ_k with multiplicity $m_j(k)$,
$s = \frac{d-1}{2} + q$ of order $d(\sigma_k)b$, where $\det C_\chi^k(\sigma_k, s)$ has a pole at $s = q$ of order b with $\mathrm{Re}(q) < 0$,

(2) *poles at*
$s = k$ of order $d_c(\chi)\, e(d,k)$, where $e(d,k) := (-1)^k(\sum_{j=0}^k (-1)^j d(\sigma_j)) \ge 0$ if $d = 2n+1$,
$s = \frac{d-1}{2}$ of order $d(\sigma_k)\frac{\mathrm{tr}(n(\sigma_k)\mathrm{Id} - C_\chi^k(\sigma_k, 0))}{2}$,
$s = \frac{d-1}{2} - q_j$ of order $d(\sigma_k)b_j$, where $\det C_\chi^k(\sigma_k, s)$ has a pole at $s = q_j$ of order b_j with $0 < q_j < \frac{d-1}{2}$,
$s = \frac{d-1}{2} - \ell$ of order $d_c(\chi)\, d(\sigma_k)$ for $\ell \in \mathbb{N} - \{\frac{d-1}{2} - k\}$ (of order $2d_c(\chi)\, d(\sigma_k)$ for $\ell \in \mathbb{N}$ if $d = 2n+1$ and $k = n$).

(3) *If $d = 2n$, $Z_\chi(\sigma_k, s)$ has the following additional zeros or poles (according to their orders are positive or negative) at*
$s = k$ of order $d_c(\chi)\, d(d,k) + (-1)^{k+1} \dim V_\chi\, E(X_\Gamma)$, where $d(d,k) := d(\sigma_k) - e(d,k) \ge 0$,
$s = -\ell$ of order $-\dim V_\chi\, E(X_\Gamma)\{\binom{2n+\ell-1}{\ell+k}\binom{\ell+k-1}{k} + \binom{2n+\ell-1}{k}\binom{2n+\ell-k-2}{\ell-1}\}$ for $\ell \in \mathbb{N}$.

Here $E(X_\Gamma)$ denotes the Euler characteristic of X_Γ. When the locations of two zeros or poles coincide, the orders of them are added.

Remark 4.7 Let $S^{2n} \simeq \mathrm{SO}(2n+1)/\mathrm{SO}(2n)$ be the $2n$-dimensional sphere, $\Delta^k_{S^{2n}}$ be the Laplacian acting on smooth k-forms $\Lambda^k(S^{2n})$ and δ be the codifferential on S^{2n}. Put $\Lambda^k_\delta(S^{2n})$ be the subspace of δ-closed forms of $\Delta^k_{S^{2n}}(\Lambda^k(S^{2n}))$. It is known that the set of all the positive eigenvalues of $\Delta^k_{S^{2n}}$ acting on $\Lambda^k_\delta(S^{2n})$ is $\{\mu_\ell = (\ell+k)(\ell+2n-k-1) | \ell \in \mathbb{N}\}$ and the multiplicity of $\mu_\ell = (\ell+k)(\ell+2n-k-1)$ is given by (See (17) of [18].)

$$\frac{(2\ell+2n-1)\,\ell(\ell+1)(\ell+2)\cdots(\ell+2n-1)}{k!(2n-k-1)!\,(\ell+k)(\ell+2n-k-1)} = \binom{2n+\ell-1}{\ell+k}\binom{\ell+k-1}{k} + \binom{2n+\ell-1}{k}\binom{2n+\ell-k-2}{\ell-1}.$$

Let us denote the μ-eigenspace by W_μ. Then we see, in the last part of Theorem 4.6, the order of $Z_\chi(\sigma_k, s)$ at $s = -\ell$ can be written as

$$-\dim V_\chi\; E(X_\Gamma)\; \dim W_{(\ell+k)(\ell+2n-k-1)}.$$

See also Theorem 4 in [22] and Chap. 3 of [20].

Remark 4.8 The proof of Theorem 4.6 is an application of the equality (4.12). One can also use the equality corresponding to $I^{k+1}_k = J^{k+1}_k$ instead of (4.12), then the zeros and poles of $Z_\chi(\sigma_k, s)$ are described in terms of $C^{k+1}_\chi(\sigma_k, s)$ instead of $C^k_\chi(\sigma_k, s)$. But the description about zeros and poles of $Z_\chi(\sigma_k, s)$ is not changed since the change from $C^{k+1}_\chi(\sigma_k, s)$ is canceled out by the one from $C_{\tau_{k+1}}(\sigma_k, s)$ by Corollary 4.3.

Proof To get a meromorphic extension of the Selberg zeta function $Z_\chi(\sigma_k, s)$, we consider meromorphic extensions of $A_{\mathrm{dis}}(s)$, $A_{\mathrm{res}}(s), \ldots,$ and $A_{\mathrm{wup}}(s)$ for $\mathrm{Re}(s) > d-1$.
Discrete spectrum: This part becomes

$$A_{\mathrm{dis}}(s) = \sum_j m_j(k) \left\{ \frac{H(i(s-\frac{d-1}{2})+\lambda_j(k))}{s-(\frac{d-1}{2}+i\lambda_j(k))} + \frac{H(i(s-\frac{d-1}{2})-\lambda_j(k))}{s-(\frac{d-1}{2}-i\lambda_j(k))} \right\}, \tag{4.15}$$

where $m_j(k)$ is the multiplicity of $\lambda_j(k)$. By using Lemma 4.5, this is a meromorphic function of s and its simple poles are located at $\frac{d-1}{2} \pm i\lambda_j(k)$ with the residues $m_j(k)$.
Residual term: This part is given by

$$d(\sigma_k) \frac{\mathrm{tr}(C^k_\chi(\sigma_k, 0) - n(\sigma_k)\mathrm{Id})}{2} \frac{H\left(i\left(s - \frac{d-1}{2}\right)\right)}{s - \left(\frac{d-1}{2}\right)}. \tag{4.16}$$

Recalling that $C^k_\chi(\sigma_k, 0)^2 = \mathrm{Id}$ and $n(\sigma_k) = 1$ unless $d = 2n+1, k = n$, and that $\mathrm{tr}(C^n_\chi(\sigma_n, 0)) = 0$ (by the fact that $\sigma_\pm$ is un-ramified) and $n(\sigma^\pm_n) = 0$. Hence we can see that the residue of this function at $s = \frac{d-1}{2}$ is always an integer.

We split $S_\chi(\sigma_k, s)$ into $C^k_\chi(\sigma_k, s)$ and $C_{\tau_k}(\sigma_k, s)$ and consider them separately.

Scattering term: We first recall that $C^k_\chi(\sigma_k, s)$ has the following form,

$$\det C^k_\chi(\sigma_k, s) = \det C^k_\chi(\sigma_k, 0)\ p^s \prod_{0<q_i<\frac{d-1}{2}} \frac{s+q_i}{s-q_i} \prod_{\mathrm{Re}(q)<0} \frac{s+\bar{q}}{s-q} \tag{4.17}$$

for some constant p. This equality can proved following the proof on p. 656 of [38] or in Section 6 of [26]. Hence we have

$$\begin{aligned} \frac{d}{ds} \log \det C^k_\chi(\sigma_k, s) = \log p + \sum_{0<q_i<\frac{d-1}{2}} \frac{1}{s+q_i} - \frac{1}{s-q_i} \\ + \sum_{\mathrm{Re}(q)<0} \frac{1}{s+\bar{q}} - \frac{1}{s-q}. \end{aligned} \tag{4.18}$$

The resulting term is given by

$$\begin{aligned} -\frac{d(\sigma_k)}{4\pi\sqrt{-1}} \int_{-\infty}^{\infty} \left(\frac{H\left(i\left(s-\frac{d-1}{2}\right)+\lambda\right)}{s-\left(\frac{d-1}{2}+i\lambda\right)} + \frac{H\left(i\left(s-\frac{d-1}{2}\right)-\lambda\right)}{s-\left(\frac{d-1}{2}-i\lambda\right)} \right) \\ \times \mathrm{tr}\left(C^k_\chi(\sigma_k, -i\lambda) \frac{d}{d\lambda} C^k_\chi(\sigma_k, i\lambda) \right) d\lambda. \end{aligned}$$

Since the integrand is an even function by (2.19), this can be reduced to

$$-\frac{d(\sigma_k)}{2\pi\sqrt{-1}} \int_{-\infty}^{\infty} \frac{H\left(i\left(s-\frac{d-1}{2}\right)+\lambda\right)}{s-\left(\frac{d-1}{2}+i\lambda\right)}\ \mathrm{tr}\left(C^k_\chi(\sigma_k, -i\lambda) \frac{d}{d\lambda} C^k_\chi(\sigma_k, i\lambda) \right) d\lambda.$$

We are now going to shift the contour of integration into the complex half plane with $\mathrm{Im}(\lambda) > 0$. Let us consider the semicircle of radius R in the upper half plane centered at 0. From (4.18) we can see that $\mathrm{tr}(C^k_\chi(\sigma_k, -i\lambda)\frac{d}{d\lambda}C^k_\chi(\sigma_k, i\lambda))$ has a simple pole at $\lambda = -iq$ within the upper half plane if $\frac{d}{ds}\log\det C^k_\chi(\sigma_k, s)$ has a simple at $s = q$ with $\mathrm{Re}(q) < 0$. Such a set of q's consists of q's with $\mathrm{Re}(q) < 0$ and $-q_j$'s with $0 < q_j < \frac{d-1}{2}$. Since the integration over the semicircle part vanishes as $R \to \infty$ by Lemma 4.5 and (4.18), using the Cauchy integral formula we can see that

$$\begin{aligned} A_{\mathrm{sct}}(s) = d(\sigma_k) \sum_{\mathrm{Re}(q)<0} b\, \frac{H\left(i\left(s-\frac{d-1}{2}-q\right)\right)}{s-\left(\frac{d-1}{2}+q\right)} \\ + d(\sigma_k) \sum_{0<q_j<\frac{d-1}{2}} (-b_j) \frac{H\left(i\left(s-\frac{d-1}{2}+q_j\right)\right)}{s-\left(\frac{d-1}{2}-q_j\right)}. \end{aligned} \tag{4.19}$$

Note that b, b_j are positive integers.

Term with the digamma function: We recall

$$\psi(1+z) \sim \log z + \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2n}}{2n\, z^{2n}} \qquad \text{as } |z| \to \infty, \tag{4.20}$$

where B_{2n} denotes the Bernoulli numbers. Using this and Fubini theorem, we see that the concerned term becomes

$$\frac{d_c(\chi)\, d(\sigma_k)}{4\pi} \int_{-\infty}^{\infty} \left(\frac{H\left(i\left(s - \frac{d-1}{2}\right) + \lambda\right)}{s - \left(\frac{d-1}{2} + i\lambda\right)} + \frac{H\left(i\left(s - \frac{d-1}{2}\right) - \lambda\right)}{s - \left(\frac{d-1}{2} - i\lambda\right)} \right)$$
$$\times \left(\psi(i\lambda + 1) + \psi(-i\lambda + 1)\right)\, d\lambda.$$

We just repeat the argument for the scattering term and obtain the following contribution

$$- d_c(\chi)\, d(\sigma_k) \sum_{\ell=1}^{\infty} \frac{H\left(i\left(s - \frac{d-1}{2} + \ell\right)\right)}{s - \frac{d-1}{2} + \ell} \tag{4.21}$$

since $\psi(z)$ has simple poles at $z = 0, -1, -2, \ldots$ with the residue -1.

Term with rational function: The term with the rational function $\frac{\frac{d-1}{2}-k}{\lambda^2+(\frac{d-1}{2}-k)^2}$ has the form

$$(-1)^{k+1} \left(\sum_{j=0}^{k-1} (-1)^j d(\sigma_j) + \sum_{j=k}^{d-1} (-1)^{j+1} d(\sigma_j) \right)$$
$$\cdot \frac{d_c(\chi)}{4\pi} \int_{-\infty}^{\infty} \left(\frac{H\left(i\left(s - \frac{d-1}{2}\right) + \lambda\right)}{s - \left(\frac{d-1}{2} + i\lambda\right)} + \frac{H\left(i\left(s - \frac{d-1}{2}\right) - \lambda\right)}{s - \left(\frac{d-1}{2} - i\lambda\right)} \right) \frac{\frac{d-1}{2} - k}{\lambda^2 + \left(\frac{d-1}{2} - k\right)^2}\, d\lambda.$$

We again repeat the same argument as above to obtain

$$(-1)^{k+1} \left(\sum_{j=0}^{k-1} (-1)^j d(\sigma_j) + \sum_{j=k}^{d-1} (-1)^{j+1} d(\sigma_j) \right) \frac{d_c(\chi)}{2} \frac{H(i(s-k))}{s-k}$$
$$= (-1)^{k+1} d_c(\chi) \left(\sum_{j=0}^{k-1} (-1)^j d(\sigma_j) \right) \frac{H(i(s-k))}{s-k}. \tag{4.22}$$

Term with even polynomial: For any even polynomial $P(\lambda)$, one can show that

$$\int_{-\infty}^{\infty} \frac{H\left(i\left(s - \frac{d-1}{2}\right) + \lambda\right)}{s - \left(\frac{d-1}{2} + i\lambda\right)} P(\lambda)\, d\lambda \equiv 0 \tag{4.23}$$

just repeating the previous argument since the even polynomial $P(\lambda)$ has no poles over $\mathbb{C}$. In particular,

$$A_{\mathrm{up}}(s) \equiv 0. \tag{4.24}$$

$A_{sct}(s) + A_{wup}(s)$: From (4.19), (4.21)–(4.23), we have

$$\begin{aligned} A_{\mathrm{sct}}(s) + A_{\mathrm{wup}}(s) = d(\sigma_k) &\sum_{\mathrm{Re}(q)<0} b\, \frac{H\left(i\left(s - \frac{d-1}{2} - q\right)\right)}{s - \left(\frac{d-1}{2} + q\right)} \\ &+ d(\sigma_k) \sum_{0<q_j<\frac{d-1}{2}} (-b_j) \frac{H\left(i\left(s - \frac{d-1}{2} + q_j\right)\right)}{s - \left(\frac{d-1}{2} - q_j\right)} \\ &- d_c(\chi)\, d(\sigma_k) \sum_{\ell=1}^{\infty} \frac{H\left(i\left(s - \frac{d-1}{2} + \ell\right)\right)}{s - \frac{d-1}{2} + \ell} \\ &+ (-1)^{k+1} d_c(\chi) \left(\sum_{j=0}^{k-1} (-1)^j d(\sigma_j)\right) \frac{H(i(s-k))}{s-k}. \end{aligned} \tag{4.25}$$

Term with Plancherel measure: If $d = 2n+1$, $\mu(\sigma_k, \lambda)$ is an even polynomial and the contribution from this is trivial by (4.23). Assume that $d = 2n$, then we have the following contribution by repeating the similar argument as before (or by the same proof on p.21 of [11] or on p. 263 of [40]):

$$A_{\mathrm{id}}(s) = -i \dim V_\chi \operatorname{Vol}(\Gamma\backslash G) \sum_j \frac{H\left(i\left(s - \frac{d-1}{2}\right) + r_j\right)}{s - \left(\frac{d-1}{2} + i r_j\right)} d_j.$$

Here, $\{r_j = i(j - \frac{1}{2}) \mid j = n-k,\ n+\ell\ (\ell \in \mathbb{N})\}$ is the set of poles of $\mu(\sigma_k, \lambda)$ in the upper half plane and d_j is the residue of $\mu(\sigma_k, \lambda)$ at $\lambda = r_j$, given by

$$d_j = \frac{d(\sigma_k)}{2^{4n-4}\Gamma(n)^2} \cdot p\left(k, i\left(j - \frac{1}{2}\right)\right).$$

We can easily calculate that

$$\begin{aligned} d_{n-k} &= \frac{d(\sigma_k)}{2^{4n-4}\,(n-1)!\,(n-1)!} \frac{i\,(-1)^{n-k-1}\,(2n-k-1)!\,k!}{2} \\ &= i\, \frac{(-1)^{n-k-1}\, n}{2^{4n-3}} \binom{2n-1}{n}, \end{aligned}$$

$$d_{n+\ell} = i\,\frac{(-1)^{n-1}\,n}{2^{4n-3}}\binom{2n-1}{n}$$
$$\cdot\left\{\binom{2n+\ell-1}{\ell+k}\binom{\ell+k-1}{k}+\binom{2n+\ell-1}{k}\binom{2n+\ell-k-2}{\ell-1}\right\}\quad(\ell\in\mathbb{N}).$$

To deal with the residue of $A_{\mathrm{id}}(s)$, we use

Proposition 4.9 *For $d = 2n$, the following equality holds*

$$\mathrm{Vol}(\Gamma\backslash G)\frac{n}{2^{4n-3}}\binom{2n-1}{n} = (-1)^n\,E(X_\Gamma). \tag{4.26}$$

Proof The proof is a refinement of the computation given in the Appendix of [10], where the equality (4.26) was derived up to a non-explicit rational number. Here we follow the computation in the Appendix of [10] taking care of this rational factor.

Let $\mathfrak{b}$ denote any subspace of $\mathfrak{g}$ with orthogonal complement $\mathfrak{b}^\perp$. Let $B_1, \ldots, B_b$ be a basis of $\mathfrak{b}$ and $B_1^*, \ldots, B_b^*$ be elements in $\mathfrak{g}^*$ such that $B_i^*(B_j) = \delta_{ij}$, $B_i^* = 0$ on $\mathfrak{b}^\perp$. Then we can define the form $\omega_\mathfrak{b}$ by

$$\omega_\mathfrak{b} = \left(\det\langle B_i, B_j\rangle\right)^{\frac{1}{2}}\,B_1^*\wedge B_2^*\wedge\ldots\wedge B_b^*,$$

where $\langle\cdot,\cdot\rangle$ is an inner product on $\mathfrak{b}^*$ induced from (2.1). The form $\omega_\mathfrak{b}$ depends only on $\mathfrak{b}$ and not on the choice of the basis $B_1, B_2, \ldots, B_b$. This consideration implies the following equality for $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$,

$$\omega_\mathfrak{g} = \omega_\mathfrak{k}\wedge\omega_\mathfrak{p},$$

where the form $\omega_\mathfrak{g}$ gives us a G-invariant volume form on G and $\omega_\mathfrak{k}$, $\omega_\mathfrak{p}$ also induce the invariant volume forms on K, G/K respectively.

From (3.5) and (3.8) in [10], we can find the following equality of the invariant form v on G induced from the normalized Haar measure in (2.3),

$$v = C\,\mathrm{Vol}(K)^{-1}\,\omega_\mathfrak{k}\wedge\omega_\mathfrak{a}\wedge\omega_\mathfrak{n}, \tag{4.27}$$

where

$$C = 2^{\frac{(d-3)}{2}}(d/2-1)!\pi^{-\frac{d}{2}}. \tag{4.28}$$

Let us remark that the normalized Haar measure is induced from (2.1), hence we have the value in (4.28), which is different from the one in (3.8) of [10]. (Note that the constant c on p. 34 of [10] is just 1 under our normalization.) From (3.12) of [10], we also have the following equality

$$\omega_\mathfrak{g} = 2^{-\frac{d-1}{2}}\,\omega_\mathfrak{k}\wedge\omega_\mathfrak{a}\wedge\omega_\mathfrak{n}. \tag{4.29}$$

By (4.27), (4.28) and (4.29), we have

$$
\begin{aligned}
v &= 2^{d-2}(d/2-1)!\pi^{-\frac{d}{2}}\mathrm{Vol}(K)^{-1}\,\omega_{\mathfrak{g}} \\
&= 2^{d-2}(d/2-1)!\pi^{-\frac{d}{2}}\mathrm{Vol}(K)^{-1}\,\omega_{\mathfrak{k}}\wedge\omega_{\mathfrak{p}}. \qquad (4.30)
\end{aligned}
$$

Now this implies

$$
\mathrm{Vol}(\Gamma\backslash G) = 2^{d-2}(d/2-1)!\pi^{-\frac{d}{2}}\mathrm{Vol}(X_\Gamma), \qquad (4.31)
$$

where $\mathrm{Vol}(X_\Gamma)$ is given by $\omega_{\mathfrak{p}}$. By Theorem 3.3 of [21], we also have

$$
\mathrm{Vol}(X_\Gamma) = (-1)^{d/2}\frac{\mathrm{Vol}(S^d)}{2}\,E(X_\Gamma), \qquad (4.32)
$$

where S^d denotes the unit sphere of dimension d. Recalling

$$
\mathrm{Vol}(S^d) = \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)} = \frac{2^{d/2+1}\pi^{d/2}}{(d-1)!!},
$$

where $(d-1)!! = (2n-1)!! = (2n-1)(2n-3)\cdots 5\cdot 3\cdot 1$, and combining (4.31) and (4.32), we obtain

$$
\mathrm{Vol}(\Gamma\backslash G) = (-1)^n 2^{3n-2}\frac{(n-1)!}{(2n-1)!!}E(X_\Gamma), \qquad (4.33)
$$

which easily implies the equality (4.26). □

By Lemma 4.5 and Proposition 4.9, $A_{\mathrm{id}}(s)$ is meromorphic on the whole $\mathbb{C}$ with simple poles with integral residues for $d = 2n$:

$$
\begin{array}{ll}
\text{poles of } A_{\mathrm{id}}(s) & \text{residues} \\
s = k & (-1)^{k+1}\dim V_\chi\, E(X_\Gamma) \\
s = -\ell\ (\ell\in\mathbb{N}) & -\dim V_\chi\, E(X_\Gamma)\{\binom{2n+\ell-1}{\ell+k}\binom{\ell+k-1}{k} + \binom{2n+\ell-1}{k}\binom{2n+\ell-k-2}{\ell-1}\}.
\end{array} \qquad (4.34)
$$

From (4.15), (4.16), (4.24), (4.25) and (4.34), we can see that $\frac{d}{ds}\log Z_\chi(\sigma_k, s)$ has a meromorphic extension to $\mathbb{C}$ with simple poles with the integer residues. Therefore we can conclude that $Z_\chi(\sigma_k, s)$ has a meromorphic extension to $\mathbb{C}$ with zeros and poles as stated above. □

4.3 Multiple gamma functions and multiple sine functions

For the functional equations of Selberg and Ruelle zeta functions, we introduce the multiple gamma function and the multiple sine function. First of all the multiple

Hurwitz zeta function is given by

$$\zeta_r(s,z) := \sum_{n_1,n_2,\ldots,n_r \geq 0} (n_1+n_2+\cdots+n_r+z)^{-s} = \sum_{n=0}^{\infty} {}_rH_n\,(n+z)^{-s},$$

where ${}_rH_n = \binom{n+r-1}{r-1}$. This series absolutely converges for $\mathrm{Re}(s) > r$. It is analytically continued to $s \in \mathbb{C}$ by the following integral expression:

$$\zeta_r(s,z) = -\frac{\Gamma(1-s)}{2\pi i}\int_C \frac{(-t)^{s-1}e^{-zt}}{(1-e^{-t})^r}\,dt.$$

The integral contour C starts at infinity on the positive real axis, circles the origin once in the positive direction excluding the points $\pm 2n\pi i$ for $n = 1, 2, \ldots$ and returns to the starting point. Therefore $\zeta_r(s,z)$ is holomorphic except for simple poles at $s = 1, \ldots, r$.

Definition 4.10 We define the multiple gamma function $\Gamma_r(z)$ by

$$\Gamma_r(z) := \exp\left(\frac{\partial}{\partial s}\,\zeta_r(s,z)|_{s=0}\right).$$

We noticed that $\Gamma_r(z)$ is a meromorphic function of order r. We recall some fundamental formulas for the multiple gamma functions. The equality ${}_rH_n = {}_rH_{n-1} + {}_{r-1}H_n$ implies that $\zeta_r(s,z) = \zeta_r(s,z+1) + \zeta_{r-1}(s,z)$. So we have the following formula.

Proposition 4.11 *The following equalities hold for $z \in \mathbb{C}$:*

$$\Gamma_r(z+1) = \Gamma_{r-1}(z)^{-1}\,\Gamma_r(z)$$

and

$$\Gamma_{n-k}(z) = \prod_{j=0}^{k} \Gamma_n(z+j)^{(-1)^j\binom{k}{j}}.$$

Definition 4.12 We define the multiple sine function $S_r(z)$ by

$$S_r(z) := \Gamma_r(z)^{-1}\Gamma_r(r-z)^{(-1)^r}.$$

If $r = 1$ then $\Gamma_1(z) = (2\pi)^{-1/2}\,\Gamma(z)$, so we have $S_1(z) = \Gamma_1(z)^{-1}\Gamma_1(1-z)^{-1} = 2\sin(\pi z)$, the usual sine function. The multiple sine function was defined and studied by Kurokawa, we recall some fundamental properties of the multiple sine functions from [23].

Proposition 4.13 *The multiple sine function $S_r(z)$ satisfies the following equalities*:

$$S_r(z+1) = S_{r-1}(z)^{-1}\, S_r(z), \quad S_{n-k}(z) = \prod_{j=0}^{k} S_n(z+j)^{(-1)^j \binom{k}{j}},$$
$$\frac{d}{dz}\log S_r(z) = (-1)^{r-1}\binom{z-1}{r-1}\pi\cot(\pi z).$$

We refer to Theorem 2.1 and 2.5 in [23] for the proof of these equalities.

4.4 Functional equation of Selberg zeta function

First we put

$$\begin{aligned}\mathcal{Z}_k(s+k) &:= \frac{d}{ds}\log Z_\chi(\sigma_k, s+k) - \Big(d_c(\chi)\, d(\sigma_k)\psi(s-d_0+k+1) \\ &\quad + \frac{d_c(\chi)\, d(d,k)}{s}\Big) \\ &= \frac{d}{ds}\log\Big(Z_\chi(\sigma_k, s+k)\Gamma(s-d_0+k+1)^{-d_c(\chi)\, d(\sigma_k)} s^{-d_c(\chi)\, d(d,k)}\Big),\end{aligned}$$

where $d_0 = \frac{d-1}{2}$. Then it is easy to see

$$\mathcal{R}_k(s) := \mathcal{Z}_k(s+k) + \mathcal{Z}_k(2d_0-k-s) + \Phi(s+k-d_0) + \mathcal{C}_k(d_0-k-s)$$

is an even entire function of s by the proof of Theorem 4.6, where

$$\Phi(s) := \dim V_\chi \operatorname{Vol}(\Gamma\backslash G)\mu(\sigma_k, is), \quad \mathcal{C}_k(s) := d(\sigma_k)\frac{d}{ds}\log\det C_\chi^k(\sigma_k, s).$$

Proposition 4.14 *The following equality holds for $s \in \mathbb{C}$,*

$$\begin{aligned}&\mathcal{Z}_k(s+k) + \mathcal{Z}_k(2d_0-k-s) + 2\hat{P}_k(i(s-d_0+k)) + \Phi(s+k-d_0) \\ &\quad = -\mathcal{C}_k(d_0-k-s).\end{aligned}$$

Here $\hat{P}_k(s) = -\frac{d_c(\chi)}{2}\widetilde{P}_k(s) + 2\pi c_\chi$, where c_χ is the constant in the unipotent term of (4.40).

Proof We let $h(s)$ be an entire function which decreases sufficiently rapidly as $|\operatorname{Im}(s)| \to \infty$ with the property $h(s) = h(2d_0-2k-s)$. We consider the contour integral

$$\mathcal{L}_T^k := \frac{1}{2\pi i}\int_{L_T^k} h(s)\mathcal{Z}_k(s+k)ds,$$

where L^k_T is the rectangle with the corners $a_1 \pm iT, a_2 \pm iT$ with $-k-\epsilon < a_1 < -k$ and $2d_0 - k < a_2 < 2d_0 - k + \epsilon$. The Cauchy integral theorem gives

$$\lim_{T\to\infty} \mathcal{L}^k_T = 2\sum_j m_j(k)h(d_0-k+i\lambda_j(k))+d(\sigma_k)\frac{\mathrm{tr}(C^k_\chi(\sigma_k,0)-n(\sigma_k)\mathrm{Id})}{2}h(d_0-k)$$
$$+ d(\sigma_k)\sum_{-d_0<\mathrm{Re}(q)<0} b\cdot h(d_0-k+q) + d(\sigma_k)$$
$$\times \sum_{0<q_j<d_0}(-b_j)\cdot h(d_0-k-q_j)$$
$$+ \delta_{d,\mathrm{even}}\,(-1)^{k+1}\dim V_\chi\, E(X_\Gamma)\,h(0). \tag{4.35}$$

Here, $\delta_{d,\mathrm{even}} = 1$ when d is even and $\delta_{d,\mathrm{even}} = 0$ otherwise. On the other hand, we have

$$\lim_{T\to\infty} \mathcal{L}^k_T = \frac{1}{2\pi i}\int_{a_1+i\infty}^{a_1-i\infty} h(s)\left(\mathcal{Z}_k(s+k)+\mathcal{Z}_k(2d_0-k-s)\right)ds$$
$$+\frac{2}{2\pi i}\int_{a_2-i\infty}^{a_2+i\infty} h(s)\mathcal{Z}_k(s+k)ds. \tag{4.36}$$

Shifting the integral line to $\mathrm{Re}(s) = d_0 - k$ for the first integral on the right hand side of (4.36),

$$\frac{1}{2\pi i}\int_{a_1+i\infty}^{a_1-i\infty} h(s)\left(\mathcal{Z}_k(s+k)+\mathcal{Z}_k(2d_0-k-s)\right)ds$$
$$=\frac{1}{2\pi i}\int_{d_0-k+i\infty}^{d_0-k-i\infty} h(s)\left(\mathcal{Z}_k(s+k)+\mathcal{Z}_k(2d_0-k-s)\right)ds$$
$$+d(\sigma_k)\sum_{-d_0<\mathrm{Re}(q)<0} b\cdot h(d_0-k+q)+d(\sigma_k)\sum_{0<q_j<d_0}(-b_j)\cdot h(d_0-k-q_j)$$
$$+\delta_{d,\mathrm{even}}\,(-1)^{k+1}\dim V_\chi\, E(X_\Gamma)\,h(0). \tag{4.37}$$

Shifting the integral line to $\mathrm{Re}(s) = d_0 - k$ for the second integral on the right hand side of (4.36),

$$\frac{1}{2\pi i}\int_{a_2-i\infty}^{a_2+i\infty} h(s)\mathcal{Z}_k(s+k)ds = \frac{1}{2\pi i}\int_{d_0-k-i\infty}^{d_0-k+i\infty} h(s)\mathcal{Z}_k(s+k)ds. \tag{4.38}$$

We combine (4.35), (4.37) and (4.38) to obtain

$$
\begin{aligned}
&2\sum_j m_j(k)h(d_0-k+i\lambda_j(k))+d(\sigma_k)\frac{\operatorname{tr}\left(C_\chi^k(\sigma_k,0)-n(\sigma_k)\mathrm{Id}\right)}{2}h(d_0-k)\\
&=\frac{2}{2\pi i}\int\limits_{d_0-k-i\infty}^{d_0-k+i\infty} h(s)\left(\mathcal{Z}_k(s+k)+\frac{1}{2}\mathcal{C}_k(d_0-k-s)+\frac{1}{2}\Phi(s+k-d_0)\right)ds\\
&\quad+\frac{1}{2\pi i}\int\limits_{d_0-k+i\infty}^{d_0-k-i\infty} h(s)\,(\mathcal{Z}_k(s+k)+\mathcal{Z}_k(2d_0-k-s)\\
&\quad+\mathcal{C}_k(d_0-k-s)+\Phi(s+k-d_0))\,ds. \qquad (4.39)
\end{aligned}
$$

The first integral on the right side of the equality (4.39) become to

$$
\begin{aligned}
&\frac{2}{2\pi}\int\limits_{-\infty}^{\infty} h(d_0-k+i\lambda)\left(\mathcal{Z}_k(d_0+i\lambda)+\frac{1}{2}\mathcal{C}_k(i\lambda)+\frac{1}{2}\Phi(i\lambda)\right)\,d\lambda\\
&\quad=2\Bigg(\frac{1}{2\pi}\sum_{\gamma\in\Gamma_{\mathrm{hyp}}}\operatorname{tr}\chi(\gamma)\,l(C_\gamma)\,j(\gamma)^{-1}\,D(\gamma)^{-1}\,\overline{\operatorname{tr}\sigma_k(m_\gamma)}\\
&\qquad\times\int\limits_{-\infty}^{\infty} h(d_0-k+i\lambda)e^{-i\lambda l(C_\gamma)}\,d\lambda\\
&\qquad-\frac{d_c(\chi)}{2\pi}\int\limits_{-\infty}^{\infty} h(d_0-k+i\lambda)\left(d(\sigma_k)\psi(i\lambda+1)+\frac{d(d,k)}{d_0-k+i\lambda}\right)\,d\lambda\\
&\qquad+\frac{1}{4\pi}\int\limits_{-\infty}^{\infty} h(d_0-k+i\lambda)\mathcal{C}_k(i\lambda)\,d\lambda\\
&\qquad+\dim V_\chi\operatorname{Vol}(\Gamma\backslash G)\frac{1}{4\pi}\int\limits_{-\infty}^{\infty} h(d_0-k+i\lambda)\,\mu(\sigma_k,\lambda)\,d\lambda\Bigg). \qquad (4.40)
\end{aligned}
$$

The equalities (4.39) and (4.40) with the equality (4.5) applied to $H^k(\lambda) := h(s)$ with $s = d_0 - k + i\lambda$ imply

$$
\frac{1}{2\pi i}\int\limits_{d_0-k+i\infty}^{d_0-k-i\infty} h(s)\left(\mathcal{R}_k(s)+2\hat{P}_k(i(s-d_0+k))\right)\,ds=0
$$

for the even polynomial $\hat{P}_k(s) = -\frac{d_c(\chi)}{2}\widetilde{P}_k(s) + 2\pi c_\chi$, where c_χ is the constant in the unipotent term of (4.40). This holds for arbitrary function $h(s)$ with the properties described before, we can conclude

$$\mathcal{Z}_k(s+k)+\mathcal{Z}_k(2d_0-k-s)+2\hat{P}_k(i(s-d_0+k))+\Phi(s+k-d_0) = -\mathcal{C}_k(d_0-k-s)$$

over the whole complex plane. □

From the above and Remark 4.1 we introduce the following notations:
For $d = 2n+1$ and $0 \le k \le n-1$, put

$$\widehat{Z}_\chi(\sigma_k, s) := Z_\chi(\sigma_k, s)\,\Gamma(s-d_0+1)^{-d_c(\chi)\,d(\sigma_k)}(s-k)^{-d_c(\chi)\,d(d,k)} \cdot \exp\left(\int_0^{s-d_0} \hat{P}_k(iz)\,dz\right) \exp\left(\frac{\dim V_\chi \operatorname{Vol}(\Gamma\backslash G)}{2}\int_0^{s-d_0} \mu(\sigma_k, iz)\,dz\right). \tag{4.41}$$

For $d = 2n+1$ and $k = n$, put

$$\widehat{Z}_\chi(\sigma_n, s) := Z_\chi(\sigma_n^+, s) Z_\chi(\sigma_n^-, s)\,\Gamma(s-d_0+1)^{-2d_c(\chi)\,d(\sigma_n)} \cdot \exp\left(2\int_0^{s-d_0} \hat{P}_n(iz)\,dz\right) \exp\left(\dim V_\chi \operatorname{Vol}(\Gamma\backslash G)\int_0^{s-d_0} \mu(\sigma_n, iz)\,dz\right). \tag{4.42}$$

For $d = 2n$ and $0 \le k \le n-1$, put

$$\widehat{Z}_\chi(\sigma_k, s) := Z_\chi(\sigma_k, s)\,\Gamma(s-d_0+1)^{-d_c(\chi)\,d(\sigma_k)}(s-k)^{-d_c(\chi)\,d(d,k)} \cdot \Gamma_d(\sigma_k, s)\, \exp\left(\int_0^{s-d_0} \hat{P}_k(iz)\,dz\right), \tag{4.43}$$

where

$$\Gamma_d(\sigma_k, s) = \left[\prod_{\ell=0}^{k} (\Gamma_d(s-\ell)\Gamma_d(s+\ell+1))^{(-1)^\ell\binom{d}{k-\ell}}\right]^{-\dim V_\chi\, E(X_\Gamma)}$$

and $\Gamma_d(s)$ is the multiple gamma function of order d. (See Definition 4.10.)

Theorem 4.15 *We have the following equality for $s \in \mathbb{C}$:*

$$\widehat{Z}_\chi(\sigma_k, 2d_0-s) = \widehat{Z}_\chi(\sigma_k, s) \cdot \det C_\chi^k(\sigma_k, s-d_0)^{d(\sigma_k)}\ \det C_\chi^k(\sigma_k, 0)^{d(\sigma_k)}. \tag{4.44}$$

Proof We get the following equality by Proposition 4.14:

$$\begin{aligned}
&\frac{d}{ds}\log\Big(Z_\chi(\sigma_k, s+k)\Gamma(s-d_0+k+1)^{-d_c(\chi)\,d(\sigma_k)}s^{-d_c(\chi)\,d(d,k)}\Big)\\
&\quad +2\hat{P}_k(i(s-d_0+k)) + \dim V_\chi\ \mathrm{Vol}(\Gamma\backslash G)\,\mu(\sigma_k, i(s+k-d_0))\\
&= \frac{d}{ds}\log\Big(Z_\chi(\sigma_k, 2d_0-k-s)\Gamma(d_0-k-s+1)^{-d_c(\chi)\,d(\sigma_k)}\\
&\quad \times\ (2(d_0-k)-s)^{-d_c(\chi)\,d(d,k)}\Big)\\
&\quad + d(\sigma_k)\frac{d}{ds}\log\ \det C_\chi^k(\sigma_k, d_0-k-s).
\end{aligned}$$

From the above and Remark 4.1, the following equality holds for $s\in\mathbb{C}$:

$$\begin{aligned}
&Z_\chi(\sigma_k, s)\,\Gamma(s-d_0+1)^{-d_c(\chi)\,d(\sigma_k)}(s-k)^{-d_c(\chi)\,d(d,k)}\\
&\quad\cdot\exp\left(\dim V_\chi\ \mathrm{Vol}(\Gamma\backslash G)\int_0^{s-d_0}\mu(\sigma_k, iz)\,dz\right)\\
&= Z_\chi(\sigma_k, 2d_0-s)\Gamma(d_0-s+1)^{-d_c(\chi)\,d(\sigma_k)}(2d_0-s-k)^{-d_c(\chi)\,d(d,k)}\\
&\quad\cdot\det C_\chi^k(\sigma_k, d_0-s)^{d(\sigma_k)}\ \det C_\chi^k(\sigma_k, 0)^{-d(\sigma_k)}\ \exp\left(-2\int_0^{s-d_0}\hat{P}_k(iz)\,dz\right)
\end{aligned}$$

unless $d=2n+1$ and $k=n$, and for this case

$$\begin{aligned}
&Z_\chi(\sigma_n^+, s)Z_\chi(\sigma_n^-, s)\,\Gamma(s-d_0+1)^{-2d_c(\chi)\,d(\sigma_n)}\\
&\quad\times\exp\left(2\dim V_\chi\ \mathrm{Vol}(\Gamma\backslash G)\int_0^{s-d_0}\mu(\sigma_n, iz)\,dz\right)\\
&= Z_\chi(\sigma_n^+, 2d_0-s)Z_\chi(\sigma_n^-, 2d_0-s)\Gamma(d_0-s+1)^{-2d_c(\chi)\,d(\sigma_n)}\\
&\quad\cdot\det\ C_\chi^n(\sigma_n, d_0-s)^{d(\sigma_n)}\ \det C_\chi^n(\sigma_n, 0)^{-d(\sigma_n)}\ \exp\left(-4\int_0^{s-d_0}\hat{P}_n(iz)\,dz\right).
\end{aligned}$$

When $d=2n$ the factor from Plancherel measure is

$$\begin{aligned}
&\exp\left(\dim V_\chi\ \mathrm{Vol}(\Gamma\backslash G)\int_0^{s-d_0}\mu(\sigma_k, iz)\,dz\right)\\
&= \exp\left(\int_0^{s-n+\frac{1}{2}}\frac{2i}{(2n-1)!}\,d(\sigma_k)\,p(k, iz)\,\pi\tan(\pi z)\,dz\right)^{(-1)^n\dim V_\chi\ E(X_\Gamma)}.
\end{aligned}$$

By Proposition 7.5 in [12], the above integral is evaluated as

$$\left[\prod_{\ell=0}^{k}\left(S_{2n}(s-\ell)\,S_{2n}(s+\ell+1)\right)^{(-1)^{\ell}\binom{2n}{k-\ell}}\right]^{\dim V_{\chi}\,E(X_{\Gamma})}$$
$$=\left[\prod_{\ell=0}^{k}\left(\frac{\Gamma_{2n}(s-\ell)}{\Gamma_{2n}(2n-s+\ell)}\frac{\Gamma_{2n}(s+\ell+1)}{\Gamma_{2n}(2n-s-\ell-1)}\right)^{(-1)^{\ell}\binom{2n}{k-\ell}}\right]^{-\dim V_{\chi}\,E(X_{\Gamma})}.$$

The rest is clear. □

5 Ruelle zeta functions

5.1 Meromorphic extension of Ruelle zeta function

The Ruelle zeta function $R_{\chi}(s)$ is defined by

$$R_{\chi}(s):=\prod_{\{\gamma\}_{\Gamma}\in \mathrm{P}\Gamma_{\mathrm{hyp}}}\det\left(\mathrm{Id}-\chi(\gamma)e^{-s\,l(C_{\gamma})}\right)^{-1}$$

for $\mathrm{Re}(s)>d-1$. Here the determinant denoted by det is taken over the representation space V_{χ} of χ, and $l(C_{\gamma})$ denotes the length of the prime geodesic C_{γ} determined by γ.

Let us recall the following equality which holds for $\mathrm{Re}(s)>d-1$,

$$R_{\chi}(s)=\prod_{k=0}^{d-1}Z_{\chi}(\sigma_k,s+k)^{(-1)^{k+1}}, \tag{5.1}$$

where $Z_{\chi}(\sigma_n,s+n)=Z_{\chi}(\sigma_n^{+},s+n)Z_{\chi}(\sigma_n^{-},s+n)$ when $d=2n+1$. Combining this equality and Theorem 4.6, we can easily obtain

Theorem 5.1 *The Ruelle zeta function $R_{\chi}(s)$ defined a priori for* $\mathrm{Re}(s)>d-1$ *has a meromorphic extension to $\mathbb{C}$.*

Proof of Theorem 1.2 If $d=2n+1$, we have

$$R_{\chi}(s)=\prod_{k=0}^{n-1}\left[Z_{\chi}(\sigma_k,s+k)\,Z_{\chi}(\sigma_k,s+2n-k)\right]^{(-1)^{k+1}}$$
$$\cdot\left(Z_{\chi}(\sigma_n^{+},s+n)Z_{\chi}(\sigma_n^{-},s+n)\right)^{(-1)^{n+1}}. \tag{5.2}$$

We see that the orders of zeros or poles of $Z_{\chi}(\sigma_k,s)$ at $s=k$ and $s=2n-k$ may contribute to the order N_0 by the above expression of $R_{\chi}(s)$. Let $m(k)$ be the number of j's such that $\lambda_j(k)^2 + (n-k)^2 = 0$ for $0 \le k \le n$, that is, the number of the zero eigenvalues of $\pi_{\sigma_k,\lambda}(\Delta_k)$ by Theorem 4.6. Since $m(k) + m(k-1) = \beta_k$, we have

$$m(k) = \beta_k - \beta_{k-1} + \cdots + (-1)^k \beta_0.$$

Hence, noting $e(d, n) = 0$ we have

$$\begin{aligned} N_0 &= -2\sum_{k=0}^{n}(-1)^{k+1}\left(\beta_k - \beta_{k-1} + \cdots + (-1)^k\beta_0\right) + \sum_{k=0}^{n-1}(-1)^{k+1}d(\sigma_k)b_k \\ &\quad + d_c(\chi)\sum_{k=0}^{n-1}(-1)^{k+1}e(d,k) \\ &= 2\sum_{k=0}^{n}(-1)^k(n+1-k)\beta_k + \sum_{k=0}^{n-1}(-1)^{k+1}d(\sigma_k)b_k + d_c(\chi)\,(-1)^n\binom{2n-2}{n-1}. \end{aligned}$$

Here, we used the identity:

$$\sum_{k=0}^{n-1}(-1)^{k+1}\left\{\sum_{j=0}^{k}(-1)^j\binom{2n}{j}\right\} = -\sum_{k=0}^{n-1}(-1)^k\binom{2n-1}{k} = (-1)^n\binom{2n-2}{n-1}.$$

If $d = 2n$, we have

$$R_\chi(s) = \prod_{k=0}^{n-1}\left[Z_\chi(\sigma_k, s+k)\,Z_\chi(\sigma_k, s+2n-1-k)^{-1}\right]^{(-1)^{k+1}}. \tag{5.3}$$

We see that the orders of zeros or poles of $Z_\chi(\sigma_k, s)$ at $s = k$ and $s = 2n-k-1$ may contribute to the order N_0 by the above expression of $R_\chi(s)$. Hence, we have

$$\begin{aligned} N_0 &= -\sum_{k=0}^{n-1}(-1)^{k+1}\cdot(-1)^{k+1}\dim V_\chi\; E(X_\Gamma) + \sum_{k=0}^{n-1}(-1)^{k+1}d(\sigma_k)b_k \\ &\quad - d_c(\chi)\sum_{k=0}^{n-1}(-1)^{k+1}d(d,k) \\ &= -n\,\dim V_\chi\; E(X_\Gamma) + \sum_{k=0}^{n-1}(-1)^{k+1}d(\sigma_k)b_k + d_c(\chi)\,(-1)^{n-1}\binom{2n-3}{n-2}. \end{aligned}$$

This completes the proof of Theorem 1.2. □

5.2 Functional equation of Ruelle zeta function

We can derive the functional equation of $R_\chi(s)$ from Theorem 4.15.

Theorem 5.2 *When $d = 2n+1$, the Ruelle zeta function $R_\chi(s)$ satisfies the following functional equation,*

$$R_\chi(-s)\,R_\chi(s)^{-1} = \exp\left((-1)^{n-1}\,4(n+1)\binom{2n}{n}^{-1}s\right)^{\dim V_\chi \operatorname{Vol}(\Gamma\backslash G)} \times Y(d,s)^{d_c(\chi)}C_\chi(d,s)\exp\left(-Q_\chi(d,s)\right).$$

Here

$$Y(d,s) := Y_1(d,s)\,Y_2(d,s),$$

where

$$Y_1(d,s) := \prod_{k=0}^{n-1}\left(\frac{s+(n-k)}{s-(n-k)}\right)^{(-1)^k a(d,k)},$$
$$Y_2(d,s) := \prod_{k=0}^{n-1}\left(\frac{s+2(n-k)}{s-2(n-k)}\right)^{(-1)^k d(d,k)}$$

with $a(d,k) := 2e(d,k) - d(\sigma_k) = \frac{n-k}{n}\,d(\sigma_k)$ *and*

$$C_\chi(d,s) := \prod_{k=0}^{n}\left(\det\,\widetilde{C}_\chi(\sigma_k,s)\right)^{(-1)^k d(\sigma_k)}$$

with

$$\widetilde{C}_\chi(\sigma_k,s) := C_\chi^k(\sigma_k,n-k-s)\,C_\chi^k(\sigma_k,-(n-k)-s)$$

for $0 \le k \le n-1$ *and*

$$\widetilde{C}_\chi(\sigma_n,s) := C_\chi^n(\sigma_n,-s)\,C_\chi^n(\sigma_n,0),$$

and

$$Q_\chi(d,s) := \sum_{k=0}^{n}(-1)^k\int_{-s}^{s}2\hat{P}_k(i(z-n+k))\,dz.$$

Proof By Theorem 4.15, we have

$$
\begin{aligned}
&Z_\chi(\sigma_k, 2d_0 - s - k)\, Z_\chi(\sigma_k, s+k)^{-1} \\
&= \left(\frac{\Gamma(d_0 - s - k + 1)}{\Gamma(s + k - d_0 + 1)}\right)^{d_c(\chi)\, d(\sigma_k)} \left(\frac{2d_0 - s - 2k}{s}\right)^{d_c(\chi)\, d(d,k)} \\
&\quad \cdot \det C_\chi^k(\sigma_k, s - d_0 + k)^{d(\sigma_k)}\ \det C_\chi^k(\sigma_k, 0)^{d(\sigma_k)} \\
&\quad \cdot \exp\left(\dim\chi\ \mathrm{Vol}(\Gamma\backslash G) \int_0^{s+k-d_0} \mu(\sigma_k, iz)\, dz\right) \exp\left(2 \int_0^{s+k-d_0} \hat{P}_k(iz)\, dz\right)
\end{aligned} \tag{5.4}
$$

unless $d = 2n + 1$ and $k = n$, and the similar functional equation for $Z_\chi(\sigma_n^+, s)$ $Z_\chi(\sigma_n^-, s)$. By (5.2) and (5.4),

$$
\begin{aligned}
R_\chi(-s)\, R_\chi(s)^{-1} &= \prod_{k=0}^{n-1} \left[\frac{Z_\chi(\sigma_k, 2n - s - k)}{Z_\chi(\sigma_k, s + k)} \frac{Z_\chi(\sigma_k, -s + k)}{Z_\chi(\sigma_k, 2n + s - k)}\right]^{(-1)^{k+1}} \\
&\quad \cdot \left[\frac{Z_\chi(\sigma_n^+, n - s) Z_\chi(\sigma_n^-, n - s)}{Z_\chi(\sigma_n^+, n + s) Z_\chi(\sigma_n^-, n + s)}\right]^{(-1)^{n+1}} \\
&= Y_1(d, s)^{d_c(\chi)}\, Y_2(d, s)^{d_c(\chi)}\, C_\chi(d, s)\, \exp(-Q_\chi(d, s)) \\
&\quad \times \varphi_d(s)^{\dim V_\chi\, \mathrm{Vol}(\Gamma\backslash G)}.
\end{aligned}
$$

The five functions on the right hand side of the last formula are given as follows. First rational factor:

$$
\begin{aligned}
Y_1(d, s) &:= \prod_{k=0}^{n} \left[\frac{\Gamma(s - n + k + 1)}{\Gamma(-s + n - k + 1)} \frac{\Gamma(s + n - k + 1)}{\Gamma(-s - n + k + 1)}\right]^{(-1)^k\, d(\sigma_k)} \\
&= \prod_{k=0}^{n} \left[\frac{\Gamma(s)^2}{\Gamma(-s)^2} \prod_{j=1}^{n-k-1} \left(\frac{s + j}{s - j}\right)^2 \cdot \frac{s + (n - k)}{s - (n - k)}\right]^{(-1)^k\, d(\sigma_k)} \\
&= \prod_{k=0}^{n-1} \left(\frac{s + (n - k)}{s - (n - k)}\right)^{(-1)^k\, a(d,k)}
\end{aligned}
$$

and the index $(-1)^k\, a(d, k)$ is calculated as

$$
2\sum_{j=0}^{k} (-1)^j d(\sigma_j) - (-1)^k d(\sigma_k) = (-1)^k \{2e(d, k) - d(\sigma_k)\} = (-1)^k\, \frac{n - k}{n}\, d(\sigma_k).
$$

Second rational factor:

$$
\begin{aligned}
Y_2(d,s) &:= \prod_{k=0}^{n-1}\left(\frac{s}{2n-2k-s}\,\frac{2n-2k+s}{-s}\right)^{(-1)^k d(d,k)} \\
&= \prod_{k=0}^{n-1}\left(\frac{s+2(n-k)}{s-2(n-k)}\right)^{(-1)^k d(d,k)}.
\end{aligned}
$$

Scattering factor:

$$
\begin{aligned}
C_\chi(d,s) &:= \prod_{k=0}^{n-1}\left[\frac{\det C_\chi^k(\sigma_k, n-s-k)}{\det C_\chi^k(\sigma_k,0)}\,\frac{\det C_\chi^k(\sigma_k,0)}{\det C_\chi^k(\sigma_k, n+s-k)}\right]^{(-1)^k d(\sigma_k)} \\
&\quad \cdot \left[\frac{\det C_\chi^n(\sigma_n,-s)}{\det C_\chi^n(\sigma_n,0)}\right]^{(-1)^n d(\sigma_n)} \\
&= \prod_{k=0}^{n-1}\det\left\{C_\chi^k(\sigma_k,-s+n-k)\,C_\chi^k(\sigma_k,-s-n+k)\right\}^{(-1)^k d(\sigma_k)} \\
&\quad \cdot \det\left\{C_\chi^n(\sigma_n,-s)\,C_\chi^n(\sigma_n,0)\right\}^{(-1)^n d(\sigma_n)}.
\end{aligned}
$$

Polynomial factor in the exponential function:

$$
\begin{aligned}
Q_\chi(d,s) &:= \sum_{k=0}^{n-1}(-1)^k\int_0^{s+k-d_0} 2\hat{P}_k(iz)\,dz - \sum_{k=0}^{n-1}(-1)^k\int_0^{-s+k-d_0} 2\hat{P}_k(iz)\,dz \\
&\quad + (-1)^n\int_0^{s+n-d_0} 4\hat{P}_n(iz)\,dz. \\
&= \sum_{k=0}^{n-1}(-1)^k\int_{-s}^{s} 2\hat{P}_k(i(z+k-n))\,dz \;+ (-1)^n\int_{-s}^{s} 2\hat{P}_n(iz)\,dz.
\end{aligned}
$$

Plancherel factor:

$$
\begin{aligned}
\varphi_d(s) &:= \exp\left(\sum_{k=0}^{n}(-1)^{k+1}\int_{-s+k-n}^{s+k-n}\mu(\sigma_k, iz)\,dz\right) \\
&= \exp\left(\sum_{k=0}^{n}(-1)^{k+1}\int_{-s}^{s}\mu\left(\sigma_k, i(z-n+k)\right)\,dz\right).
\end{aligned}
$$

Then we have

$$\frac{d}{ds}\log\varphi_d(s) = F_n(s) + F_n(-s),$$

where $F_n(z)$ is the polynomial defined by

$$F_n(z) = \sum_{k=0}^{n}(-1)^{k+1}\mu\left(\sigma_k, i(z-n+k)\right).$$

By Theorem 4.4 in [3], $F_n(s)+F_n(-s)$ is a constant function. By the explicit formula of $\mu(\sigma_k,\lambda)$, we can see that

$$\mu(\sigma_k, i(-n+k)) = \frac{4n!n!}{(2n)!(2n)!}d(\sigma_k)\frac{(-1)^{n+k}}{2}k!(2n-k)! \qquad (0\le k\le n-1),$$
$$\mu(\sigma_n, i(-n+n)) = \frac{4n!n!}{(2n)!(2n)!}d(\sigma_n)\, n!\,n!.$$

Thus we have

$$F_n(0) = \sum_{k=0}^{n}(-1)^{k+1}\mu\left(\sigma_k, i(-n+k)\right) = (-1)^{n-1}(2n+2)\binom{2n}{n}^{-1}.$$

Finally we have

$$\varphi_d(s) = \exp\left(\int_{-s}^{s}F_n(z)\,dz\right) = \exp\left(2F_n(0)\,s + C_n\right)$$

for some constant C_n. Put $s=0$ then we have $C_n\equiv 0$. □

Theorem 5.3 *When $d=2n$, the Ruelle zeta function $R_\chi(s)$ satisfies the following functional equation,*

$$R_\chi(-s)\,R_\chi(s) = (-1)^{n\dim V_\chi\,E(X_\Gamma)}\,(2\sin(\pi s))^{d\,\dim V_\chi\,E(X_\Gamma)}\,Y(d,s)^{d_c(\chi)} \times C_\chi(d,s)\exp\left(-Q_\chi(d,s)\right).$$

Here

$$Y(d,s) := Y_1(d,s)\,Y_2(d,s),$$

where

$$Y_1(d,s) := (-1)^n\prod_{k=0}^{n-1}\left(\left(s+\left(n-k-\frac{1}{2}\right)\right)\left(s-\left(n-k-\frac{1}{2}\right)\right)\right)^{(-1)^k a(d,k)}$$

with $a(d,k) := 2e(d,k) - d(\sigma_k) = \frac{2n-1-2k}{2n-1} d(\sigma_k)$ *and*

$$Y_2(d,s) := s^{2\alpha(n)} \prod_{k=0}^{n-1} \left(\left(s + 2\left(n-k-\frac{1}{2}\right)\right)\left(s - 2\left(n-k-\frac{1}{2}\right)\right)\right)^{(-1)^k d(d,k)}$$

with $\alpha(n) = (-1)^n \binom{2n-3}{n-2}$ *and*

$$C_\chi(d,s) := \prod_{k=0}^{n-1} \det\left(\widetilde{C}_\chi(\sigma_k,s)\right)^{(-1)^k d(\sigma_k)}$$

with

$$\widetilde{C}_\chi(\sigma_k,s) := C_\chi^k\left(\sigma_k, s - \left(n-k-\frac{1}{2}\right)\right) C_\chi^k\left(\sigma_k, -s - \left(n-k-\frac{1}{2}\right)\right)$$

and

$$Q_\chi(d,s) := \sum_{k=0}^{n-1} (-1)^k \int_{s-(n-k-\frac{1}{2})}^{s+(n-k-\frac{1}{2})} 2\hat{P}_k(iz)\,dz.$$

Proof By Theorem 4.15, we have

$$\begin{aligned}
&Z_\chi(\sigma_k, 2d_0 - s - k)\, Z_\chi(\sigma_k, s+k)^{-1} \\
&\quad= \left(\frac{\Gamma(d_0 - s - k + 1)}{\Gamma(s + k - d_0 + 1)}\right)^{d_c(\chi)\, d(\sigma_k)} \left(\frac{2d_0 - s - 2k}{s}\right)^{d_c(\chi)\, d(d,k)} \\
&\qquad\cdot \det C_\chi^k(\sigma_k, s - d_0 + k)^{d(\sigma_k)} \ \det C_\chi^k(\sigma_k, 0)^{d(\sigma_k)} \\
&\qquad\cdot \exp\left(\dim V_\chi \operatorname{Vol}(\Gamma\backslash G) \int_0^{s+k-d_0} \mu(\sigma_k, iz)\,dz\right) \exp\left(2\int_0^{s+k-d_0} \hat{P}_k(iz)\,dz\right). \qquad (5.5)
\end{aligned}$$

By (5.3) and (5.5),

$$\begin{aligned}
R_\chi(-s)R_\chi(s) &= \prod_{k=0}^{n-1}\left[\frac{Z_\chi(\sigma_k, 2n-1-s-k)}{Z_\chi(\sigma_k, s+k)} \frac{Z_\chi(\sigma_k, 2n-1+s-k)}{Z_\chi(\sigma_k, -s+k)}\right]^{(-1)^k} \\
&= Y_1(d,s)^{d_c(\chi)} Y_2(d,s)^{d_c(\chi)} \det C_\chi(d,s) \exp(-Q_\chi(d,s)) \varphi_d(s)^{\dim V_\chi}.
\end{aligned}$$

The five functions on the right hand side of the last formula are given as follows.

First rational factor:

$$Y_1(d,s) := \prod_{k=0}^{n-1}\left[\frac{\Gamma(-s+n-k+\frac{1}{2})}{\Gamma(s-n+k+\frac{3}{2})}\,\frac{\Gamma(s+n-k+\frac{1}{2})}{\Gamma(-s-n+k+\frac{3}{2})}\right]^{(-1)^k d(\sigma_k)}$$
$$= \prod_{k=0}^{n-1}\left[\frac{-1}{(s+\frac{1}{2}-n+k)(s-\frac{1}{2}+n-k)}\prod_{j=0}^{n-k-1}\left(\left(s-\frac{1}{2}-j\right)\left(s+\frac{1}{2}+j\right)\right)^2\right]^{(-1)^k d(\sigma_k)}$$
$$= (-1)^n\prod_{k=0}^{n-1}\left(\left(s-\frac{1}{2}+(n-k)\right)\left(s+\frac{1}{2}-(n-k)\right)\right)^{(-1)^k a(d,k)}$$

and the index $(-1)^k a(d,k)$ is calculated as

$$2\sum_{j=0}^{k}(-1)^j d(\sigma_j)-(-1)^k d(\sigma_k)=(-1)^k\{2e(d,k)-d(\sigma_k)\}=(-1)^k\frac{2n-1-2k}{2n-1}d(\sigma_k).$$

Second rational factor:

$$Y_2(d,s) := \prod_{k=0}^{n-1}\left(\frac{2n-1-s-2k}{s}\,\frac{2n-1+s-2k}{-s}\right)^{(-1)^k d(d,k)}$$
$$= s^{2\alpha(n)}\prod_{k=0}^{n-1}\left((s-1+2(n-k))\,(s+1-2(n-k))\right)^{(-1)^k d(d,k)}$$

with

$$\alpha(n) := -\sum_{k=0}^{n-1}(-1)^k d(d,k) = \sum_{k=1}^{n-1}\sum_{j=0}^{k-1}(-1)^j d(\sigma_j)$$
$$= \sum_{k=0}^{n-2}(-1)^k(n-1-k)\,d(\sigma_k) = (-1)^n\binom{2n-3}{n-2}.$$

Scattering factor:

$$C_\chi(d,s) := \prod_{k=0}^{n-1}\Bigg[\det C_\chi^k\left(\sigma_k, s+\frac{1}{2}-n+k\right)\det C_\chi^k(\sigma_k,0)$$
$$\cdot\det C_\chi^k\left(\sigma_k,-s+\frac{1}{2}-n+k\right)\det C_\chi^k(\sigma_k,0)\Bigg]^{(-1)^k d(\sigma_k)}$$
$$= \prod_{k=0}^{n-1}\det\left\{C_\chi^k\left(\sigma_k, s+\frac{1}{2}-n+k\right)C_\chi^k\left(\sigma_k,-s+\frac{1}{2}-n+k\right)\right\}^{(-1)^k d(\sigma_k)}.$$

Polynomial factor in the exponential function:

$$Q_\chi(d,s) := \sum_{k=0}^{n-1}(-1)^{k+1}\int_0^{s+k-d_0} 2\hat{P}_k(iz)\,dz + \sum_{k=0}^{n-1}(-1)^{k+1}\int_0^{-s+k-d_0} 2\hat{P}_k(iz)\,dz$$
$$= \sum_{k=0}^{n-1}(-1)^{k}\int_{s-n+k+\frac{1}{2}}^{s+n-k-\frac{1}{2}} 2\hat{P}_k(iz)\,dz.$$

Plancherel factor:

$$\varphi_d(s) := \prod_{k=0}^{n-1}\left[\frac{\Gamma_{2n}(\sigma_k, s+k)}{\Gamma_{2n}(\sigma_k, -s+2n-k)}\,\frac{\Gamma_{2n}(\sigma_k, -s+k)}{\Gamma_{2n}(\sigma_k, s+2n-k)}\right]^{(-1)^{k+1}E(X_\Gamma)}$$
$$= \prod_{k=0}^{n-1}\left[\prod_{\ell=0}^{k}\left(\frac{\Gamma_{2n}(2n-s-k+l)}{\Gamma_{2n}(s+k-l)}\,\frac{\Gamma_{2n}(2n-s-k-l-1)}{\Gamma_{2n}(s+k+l+1)}\right)^{(-1)^\ell\binom{2n}{k-\ell}}\right]^{(-1)^k E(X_\Gamma)}$$
$$\cdot\prod_{k=0}^{n-1}\left[\prod_{\ell=0}^{k}\left(\frac{\Gamma_{2n}(2n+s-k+l)}{\Gamma_{2n}(-s+k-l)}\,\frac{\Gamma_{2n}(2n+s-k-l-1)}{\Gamma_{2n}(-s+k+l+1)}\right)^{(-1)^\ell\binom{2n}{k-\ell}}\right]^{(-1)^k E(X_\Gamma)}$$
$$= \prod_{k=0}^{n-1}\left[\prod_{\ell=0}^{k}\left(S_{2n}(s+k-l)\,S_{2n}(s+k+l+1)\right)^{(-1)^\ell\binom{2n}{k-\ell}}\right]^{(-1)^k E(X_\Gamma)}$$
$$\cdot\prod_{k=0}^{n-1}\left[\prod_{\ell=0}^{k}\left(S_{2n}(-s+k-l)\,S_{2n}(-s+k+l+1)\right)^{(-1)^\ell\binom{2n}{k-\ell}}\right]^{(-1)^k E(X_\Gamma)}$$
$$= \left(-4\sin^2(\pi s)\right)^{n\,E(X_\Gamma)} = (-1)^{nE(X_\Gamma)}\,(2\sin(\pi s))^{d\,E(X_\Gamma)}.$$

For the equalities in the last line, we used the formula of multiple sine functions on p. 276 of [12]. □

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