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Reidemeister torsion, complex volume and the Zograf infinite product for hyperbolic 3–manifolds

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We prove an equality which involves Reidemeister torsion, complex volume and the Zograf infinite product for hyperbolic 3–manifolds.

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1 Introduction

We prove an equality which involves Reidemeister torsion, complex volume and the Zograf infinite product for hyperbolic 3–manifolds. We also derive some contributions from cusps in this equality for hyperbolic 3–manifolds with cusps, which are given by Dedekind eta functions and theta functions.

To state the main result of this paper, let us introduce some notation. Let $\mathcal{M}_0$ denote a complete hyperbolic 3–manifold with cusps. Then we have a complex-valued invariant, called *complex volume*,

$$\mathbb{V}(\mathcal{M}_0) = \mathrm{Vol}(\mathcal{M}_0) + i2\pi^2\,\mathrm{CS}(\mathcal{M}_0) \mod i\pi^2\mathbb{Z}, \tag{1-1}$$

where the real part $\mathrm{Vol}(\mathcal{M}_0)$ denotes the hyperbolic volume of $\mathcal{M}_0$ and the imaginary part $\mathrm{CS}(\mathcal{M}_0)$ denotes the Chern–Simons invariant defined by the Levi-Civita connection of the hyperbolic metric of $\mathcal{M}_0$. The complex volume plays an important role in the research of hyperbolic 3–manifolds of finite volume, and has been studied extensively by Neumann and Zagier [26], Yoshida [37], Neumann and Yang [25] and Zickert [38].

Another main object appearing in our result is the Reidemeister torsion attached to a certain representation of $\pi_1(\mathcal{M}_0)$. This representation is defined to be the composition of the k^{th} symmetric tensor of the natural action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^2$ and an $\mathrm{SL}(2,\mathbb{C})$–lift of the holonomy representation $\rho\colon \pi_1(\mathcal{M}_0) \to \mathrm{PSL}(2,\mathbb{C})$. We denote by ρ^k the resulting representation of $\pi_1(\mathcal{M}_0)$. A choice of the $\mathrm{SL}(2,\mathbb{C})$–lifting corresponds to a spin structure on $\mathcal{M}_0$. The definition of Reidemeister torsion also involves a choice of

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basis of the homology groups $H_*(\mathcal{M}_0, \rho^k)$ in general, hence it is necessary to specify a basis of $H_*(\mathcal{M}_0, \rho^k)$ in order to define Reidemeister torsion. If $H_*(\mathcal{M}_0, \rho^k)$ is nontrivial, there is a canonical way to get a basis of $H_*(\mathcal{M}_0, \rho^k)$ from a simple closed curve c_i in the torus section T_i associated to the i^{th} cusp of $\mathcal{M}_0$ for $i = 1, \dots, h$. Here h denotes the number of cusps. Let us denote by $\mathcal{T}(\mathcal{M}_0, \rho^k, \{c_i\})$ the resulting Reidemeister torsion.

The third object appearing in our result is the *Zograf infinite product*, which was introduced by Zograf in [39]. We also refer to McIntyre and Takhtajan [17] for the Zograf infinite product. This is defined by

$$F_n(\mathcal{M}_0) = \prod_{[\gamma]_{\mathrm{p}}} \prod_{m=n}^{\infty} (1 - \mathfrak{q}_\gamma^m) \quad \text{for } n \geq 3. \tag{1-2}$$

Here the first product is taken over the set of conjugacy classes of the primitive loxodromic elements $\gamma \in \Gamma \subset \mathrm{PSL}(2, \mathbb{C})$, where Γ is the image of the holonomy representation ρ of $\pi_1(\mathcal{M}_0)$, and $\mathfrak{q}_\gamma = \exp(-(l_\gamma + i\theta_\gamma))$ with l_γ and θ_γ denoting the length and the torsion of the prime geodesic determined by the corresponding conjugacy class $[\gamma]_{\mathrm{p}}$, respectively.

In our result for hyperbolic 3–manifolds with cusps, there are also contributions from cusps, so we need to introduce some notation for these. Let us choose a pair of simple closed curves (m_i, l_i) on the torus section T_i associated to the i^{th} cusp for each i, which form a basis of $H_1(T_i, \mathbb{Z})$. The holonomy representation ρ induces a representation of the subgroup of $\pi_1(\mathcal{M}_0)$ generated by (m_i, l_i), which determines a complex numbers τ_i with $\mathrm{Im}(\tau_i) > 0$ for $i = 1, \dots, h$. Then τ_i is the modulus of the Euclidean structure on T_i with respect to (m_i, l_i) (see Section 6.1).

The main result of this paper is the following theorem:

Theorem 1.1 *Let $\mathcal{M}_0$ be a complete hyperbolic 3–manifold of finite volume with h cusps. For $n \geq 3$,*

$$\left| \mathcal{T}(\mathcal{M}_0, \rho^{2(n-1)}, \{m_i\}) \prod_{i=1}^{h} \eta(\tau_i)^2 \right|^{-1} = \left| \exp\Bigl(\frac{1}{\pi}\bigl(n^2 - n + \tfrac{1}{6}\bigr)\mathbb{V}(\mathcal{M}_0)\Bigr) F_n(\mathcal{M}_0) \right|, \tag{1-3}$$

where $\eta(\tau_i)$ denotes the Dedekind eta function of τ_i.

Some remarks for the equality (1-3) are in order. First, when $k = 2(n-1)$, the representation ρ^k does not depend on the choice of the lifting to $\mathrm{SL}(2, \mathbb{C})$ since the representation $\rho^{2(n-1)}$ factors through $\mathrm{PSL}(2, \mathbb{C})$. Hence $\mathcal{T}(\mathcal{M}_0, \rho^{2(n-1)}, \{m_i\})$ does not depend on this choice. Secondly, the left-hand side of the equality (1-3) does not depend on a choice of a basis (m_i, l_i) of $H_1(T_i, \mathbb{Z})$ since a basis change will also cause a change of τ_i. As we can see from its proof in Section 7, these changes cancel each other.

A corresponding equality to (1-3) for ρ^{2n-1} is also given in Theorem 8.1, where the contributions from cusps are given in terms of the Dedekind eta function and a theta function. It seems to be desirable to obtain equalities which hold as a relationship between complex-valued invariants. These refined results without modulus signs for compact hyperbolic 3–manifolds are given in Theorems 5.2 and 5.3. For the noncompact case with cusps, some parts of the proofs used for the compact case do not work, and we are able to prove only the equalities as relationships between absolute values of complex-valued invariants.

For a hyperbolic 3–manifold with cusps $\mathcal{M}_0$, by the fundamental work of Thurston [34], there exists a deformation space $\mathscr{D}(\mathcal{M}_0)$ of (in)complete hyperbolic structures on the underlying topological manifold of $\mathcal{M}_0$. Let us denote by $\mathcal{M}_u$ the corresponding (in)complete hyperbolic 3–manifold for each point $u \in \mathscr{D}(\mathcal{M}_0)$. There is a corresponding holonomy representation $\rho_u\colon \pi_1(\mathcal{M}_u) \to \mathrm{PSL}(2,\mathbb{C})$, for which one can define a representation ρ_u^k of $\pi_1(\mathcal{M}_u)$ as before. From the definitions of Reidemeister torsion and Zograf infinite product, which depend on (in)complete hyperbolic structures through ρ_u, one can see that these invariants extend to be holomorphic functions over a small open neighborhood V of the origin $\mathcal{M}_0$ in $\mathscr{D}(\mathcal{M}_0)$. We refer to Section 6.2 for the Zograf infinite product for $\mathcal{M}_u$ with $u \in V$. By [37], which is explained briefly in Section 6.2, the complex volume $\mathbb{V}(\mathcal{M}_0)$ also has an extension $\pi^2 f(u)$ over V in $\mathscr{D}(\mathcal{M}_0)$ such that $\exp(2\pi f)$ is holomorphic over V. These facts and Theorems 1.1 and 5.2 lead the author to make the following conjecture:

Conjecture 1.2 There exists an open neighborhood V of the origin in $\mathscr{D}(\mathcal{M}_0)$ where, for $n \geq 3$,

$$\mathcal{T}(\mathcal{M}_u, \rho_u^{2(n-1)}, \{m_i\})^{-12} \prod_{i=1}^{h} \eta(\tau_i(u))^{-24} \\ = c_{\mathcal{M}_0,n} \exp(2(6n^2-6n+1)\pi f(u)) F_n(\mathcal{M}_u)^{12},$$

where $\mathcal{M}_u$ denotes the (in)complete hyperbolic 3–manifold corresponding to $u \in V$ and $c_{\mathcal{M}_0,n}$ is a constant depending only on $\mathcal{M}_0$ and n with $|c_{\mathcal{M}_0,n}| = 1$.

Let us remark that we need to take the 12^{th} power of the equality (1-3) to have well-defined complex functions over $V \subset \mathscr{D}(\mathcal{M}_0)$ (see also Theorem 5.2). The equality conjectured is also suggested by the main results in McIntyre and Takhtajan [17] and McIntyre and Park [16], where similar equalities are proved for some convex cocompact hyperbolic 3–manifolds.

Now let us explain the structure of this paper. This paper is a combined version of two preprints [29; 30], so this paper consists of two parts, which deal with the compact and noncompact cases, respectively. Part I consists of Sections 2–5, and Part II consists of Sections 6–8. In Section 2, we review some basic material which is needed for the proof for the compact case. In Section 3, we introduce various zeta functions and prove a primitive version of main theorems for the compact case. In Section 4, we prove equalities between invariants derived from the Selberg trace formulas, and we review the work of Cappell and Miller [7]. In Section 5, we prove the main theorems for the compact case by combining all the results developed in the prior sections. In Section 6, we review some basic material which is needed for the proof for the noncompact case. In Sections 7 and 8, we prove Theorems 1.1 and 8.1.

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Part I Compact case

The purpose of Part I is to prove Theorems 5.2 and 5.3, which state the equalities as relationships between complex-valued invariants — Reidemeister torsion, complex volume and the Zograf infinite product — for compact hyperbolic 3–manifolds.

2 Basic materials, I

2.1 Hyperbolic 3–space as a symmetric space

Let $G = \mathrm{SL}(2,\mathbb{C})$ and $K = \mathrm{SU}(2)$ be a maximal compact subgroup of G. Recall that G is a double cover of $\mathrm{PSL}(2,\mathbb{C})$, which is the isometry group of the hyperbolic

3–space $\mathbb{H}^3$. The action of $\mathrm{PSL}(2,\mathbb{C})$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} q = \frac{aq+b}{cq+d}, \tag{2-1}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}(2,\mathbb{C})$ and $q = z + tj$ is the quaternion representation of a point $(z,t) \in \mathbb{H}^3$. Therefore G also acts on $\mathbb{H}^3$ and the isotropy subgroup of $(0,1) \in \mathbb{H}^3$ is $\mathrm{SU}(2)$, hence $\mathbb{H}^3 \cong G/K$. From now on, we use this realization of the hyperbolic 3–space as a symmetric space to apply some basic harmonic analysis over a symmetric space.

Let $G = NAK$ be the Iwasawa decomposition of G, where

$$N = \left\{ \begin{pmatrix} 1 & x+iy \\ 0 & 1 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}, \quad A = \left\{ \begin{pmatrix} e^u & 0 \\ 0 & e^{-u} \end{pmatrix} \;\middle|\; u \in \mathbb{R} \right\}. \tag{2-2}$$

We assume that the Haar measures of N, A and K are given by $dn = dx\,dy$, du and dk, respectively, where dk has total mass 1. A Cartan subgroup T of G is given by AM, where

$$M = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \;\middle|\; \theta \in [0, 2\pi] \right\}. \tag{2-3}$$

We take the Haar measure of T to be $dt = \frac{1}{2\pi}\,du\,d\theta$. The set of unitary characters of M, denoted by $\widehat{M}$, is parametrized by $k \in \mathbb{Z}$. The character corresponding to k is given by

$$\sigma_k\left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\right) = e^{ik\theta}. \tag{2-4}$$

Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{n}$, $\mathfrak{a}$ and $\mathfrak{m}$ be the Lie algebras of G, K, N, A and M, respectively. Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{2-5}$$

be the Cartan decomposition of $\mathfrak{g}$ given by the Cartan involution θ, where $\mathfrak{k}$ and $\mathfrak{p}$ are the 1 and -1 eigenspaces of θ, respectively. Let α be the unique positive root of $(\mathfrak{g},\mathfrak{a})$. Let $H \in \mathfrak{a}$ be such that $\alpha(H) = 1$. Let $\mathfrak{a}^+ \subset \mathfrak{a}$ be the positive Weyl chamber and $A^+ = \exp(\mathfrak{a}^+)$. Put $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{m}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.

The Cartan–Killing form C is positive definite on $\mathfrak{p}$ and is negative definite on $\mathfrak{k}$. We may identify $\mathfrak{p}$ with the tangent space to G/K at the identity coset. Then C provides an invariant metric on G/K. We use a normalized symmetric bilinear form defined by

$$C_0(X,Y) = \tfrac{1}{4}C(X,Y) \quad \text{for } X, Y \in \mathfrak{g}, \tag{2-6}$$

so that the corresponding invariant metric has constant curvature -1. Let $\{Z_i\}$ be an orthonormal basis for $\mathfrak{k}$ with respect to $-C_0$ and $\{Z_j\}$ be an orthonormal basis for $\mathfrak{p}$ with respect to C_0. Then the normalized Casimir elements Ω and Ω_K in the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{k}_{\mathbb{C}}$ are given by

$$\Omega = -\sum_i Z_i^2 + \sum_j Z_j^2, \quad \Omega_K = -\sum_i Z_i^2. \tag{2-7}$$

For the right regular representation R of G on $C^\infty(G)$ defined by $R(g_2)f(g_1) = f(g_1g_2)$ for $g_1, g_2 \in G$ and $f \in C^\infty(G)$, the normalized Casimir element induces a differential operator, denoted by $R(\Omega)$.

Let Γ be a cocompact torsion-free discrete subgroup of G. Then $\mathcal{M}_\Gamma := \Gamma\backslash\mathbb{H}^3$ is a compact hyperbolic manifold by definition and this is a special case of a locally symmetric space with a realization of the double coset space $\Gamma\backslash G/K$. Any compact hyperbolic 3–manifold has such a realization. Since we assume that Γ is a discrete subgroup of $G = \mathrm{SL}(2,\mathbb{C})$ rather than $\mathrm{PSL}(2,\mathbb{C})$, the resulting manifold $\mathcal{M}_\Gamma$ is equipped with a spin structure.

For a nontrivial $\gamma \in \Gamma$, there exist $g \in G$, $a_\gamma \in A^+$ and $m_\gamma \in M$ such that

$$g\gamma g^{-1} = a_\gamma m_\gamma. \tag{2-8}$$

It is known that a_γ depends only on γ and m_γ is determined by γ up to conjugacy in M (see Lemma 6.6 of [35]). By definition, there exist $\ell_\gamma > 0$ and $\theta_\gamma \in [0, 2\pi]$ such that

$$a_\gamma = \exp(\ell_\gamma H) = \begin{pmatrix} e^{\ell_\gamma/2} & 0 \\ 0 & e^{-\ell_\gamma/2} \end{pmatrix}, \quad m_\gamma = \begin{pmatrix} e^{i\theta_\gamma/2} & 0 \\ 0 & e^{-i\theta_\gamma/2} \end{pmatrix}. \tag{2-9}$$

From (2-1), it follows that $a_\gamma m_\gamma$ acts on $\mathbb{H}^3$ by $(z,t) \to (e^{\ell_\gamma + i\theta_\gamma} z, e^{\ell_\gamma} t)$. The positive real number ℓ_γ is the length of the unique closed geodesic C_γ in $\mathcal{M}_\Gamma$ that corresponds to the conjugacy class of γ in Γ. A closed geodesic C_γ also corresponds to a fixed point of the geodesic flow on the unit sphere bundle $\Gamma\backslash G/M$ over $\mathcal{M}_\Gamma \cong \Gamma\backslash G/K$. Its tangent bundle is given by $\Gamma\backslash G \times_M (\bar{\mathfrak{n}} \oplus \mathfrak{a} \oplus \mathfrak{n})$, where $\bar{\mathfrak{n}} = \theta(\mathfrak{n})$, and M acts on $\bar{\mathfrak{n}} \oplus \mathfrak{a} \oplus \mathfrak{n}$ by the adjoint action Ad. The Poincaré map $P(C_\gamma)$ is the differential of the geodesic flow at C_γ, which is given by $P(C_\gamma) = \mathrm{Ad}(a_\gamma m_\gamma)$ if $\gamma = a_\gamma m_\gamma$. Now we put

$$\begin{aligned} D(\gamma) &:= \left|\det(\mathrm{Ad}(a_\gamma m_\gamma)|_{\bar{\mathfrak{n}}\oplus\mathfrak{n}} - \mathrm{Id})\right|^{1/2} = e^{-\ell_\gamma}\left|\det(\mathrm{Ad}(a_\gamma m_\gamma)|_{\mathfrak{n}} - \mathrm{Id})\right| \\ &= e^{\ell_\gamma} \det(\mathrm{Id} - \mathrm{Ad}(a_\gamma m_\gamma)|_{\bar{\mathfrak{n}}}) = e^{\ell_\gamma}(1 - e^{-(\ell_\gamma + i\theta_\gamma)})(1 - e^{-(\ell_\gamma - i\theta_\gamma)}). \end{aligned} \tag{2-10}$$

A nontrivial $\gamma \in \Gamma$ is called *primitive* if it cannot be written as $\gamma = \gamma_0^k$ for some other $\gamma_0 \in \Gamma$ and $k > 0$. For any nontrivial $\gamma \in \Gamma$, there exists a unique primitive element $\gamma_0 \in \Gamma$ and $n_\gamma \in \mathbb{N}$ such that $\gamma = \gamma_0^{n_\gamma}$.

2.2 Bundles induced by representations

By Proposition 2.2.3 of [12], for an integer $m \geq 0$, there exists a unique (up to equivalence) irreducible representation

$$\rho^m \colon G \to \mathrm{SL}(S^m(\mathbb{C}^2)), \tag{2-11}$$

which is given by the m^{th} symmetric power of the standard representation of G on $\mathbb{C}^2$. The restrictions of ρ^m to AM decomposes as

$$\rho^m|_{AM} = \bigoplus_{k=0}^{m} e^{(m/2-k)\alpha} \otimes \sigma_{m-2k}. \tag{2-12}$$

Here we use the notation α and σ_{m-2k} explained in the previous subsection.

For a finite-dimensional irreducible representation (χ, V_χ) of Γ, we define a flat vector bundle E_χ over $\mathcal{M}_\Gamma = \Gamma \backslash G/K$ by

$$E_\chi = \Gamma \backslash (G/K \times V_\chi), \tag{2-13}$$

where Γ acts on $G/K \times V_\chi$ by $\gamma(gK, v) = (\gamma gK, \chi(\gamma)v)$. In this paper, we mainly use the restriction of the representation ρ^m in (2-11) to Γ to define a flat vector bundle by (2-13). Throughout, we denote by E_{ρ^m} the resulting flat vector bundle over $\mathcal{M}_\Gamma$.

For a finite-dimensional irreducible representation (τ, V_τ) of K, we also define a locally homogeneous vector bundle E_τ over $\mathcal{M}_\Gamma$ by

$$E_\tau = (\Gamma \backslash G \times V_\tau)/K, \tag{2-14}$$

where K acts on $\Gamma \backslash G \times V_\tau$ by $(\Gamma g, v)k = (\Gamma gk, \tau(k)^{-1}v)$.

For $m \geq 0$, we denote by τ_m the irreducible representation of K given by the restriction of ρ^m to K. By (2-12),

$$\tau_m|_M = \bigoplus_{k=0}^{m} \sigma_{m-2k}. \tag{2-15}$$

Let $R(K)$ and $R(M)$ denote the representation rings of K and M, respectively. The inclusion $\iota\colon M \to K$ induces the restriction map $\iota^*\colon R(K) \to R(M)$. By (2-15),

$$\begin{aligned} \iota^*(\tau_m - \tau_{m-2}) &= \sigma_m + \sigma_{-m} \quad \text{for } m \ge 2, \\ \iota^*(\tau_1) &= \sigma_1 + \sigma_{-1}, \\ \iota^*(\tau_0) &= \sigma_0. \end{aligned} \tag{2-16}$$

Note that τ_1 is the *spin representation* of K and σ_1 and σ_{-1} are the *half-spin representations* of M.

2.3 Eta invariant and Chern–Simons invariant

The locally homogeneous vector bundle E_{τ_1} defined by the spin representation τ_1 is equipped with the Dirac operator

$$Ds = \sum_{i=1}^{3} c(e_i)\nabla^{\tau_1}_{e_i} s \quad \text{for } s \in C^\infty(\mathcal{M}_\Gamma, E_{\tau_1}), \tag{2-17}$$

where $c(e_i)$ denotes the Clifford multiplication of an local orthonormal frame $\{e_i\}$ of $T\mathcal{M}_\Gamma$ and ∇^{τ_1} denotes the unique locally G–invariant connection of E_{τ_1}. Then canonically the vector bundle $E_{\tau_1} \otimes E_{\tau_{k-1}}$ is equipped with the Dirac operator defined in a similar way to (2-17), replacing ∇^{τ_1} by $\nabla^{\tau_1} \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{\tau_{k-1}}$, where $\nabla^{\tau_{k-1}}$ denotes the unique locally G–invariant connection of $E_{\tau_{k-1}}$. We denote by $D(\sigma_k)$ the resulting Dirac operator acting on $C^\infty(\mathcal{M}_\Gamma, E_{\tau_1 \otimes \tau_{k-1}})$. The representations σ_k and τ_{k-1} are related by

$$\sigma_k - \sigma_{-k} = (\sigma_1 - \sigma_{-1}) \otimes i^*(\tau_{k-1}), \tag{2-18}$$

where $i^*\colon R(K) \to R(M)$. By the above construction, $D(\sigma_1)$ denotes the Dirac operator defined by a spin structure, and $D(\sigma_2)$ denotes the odd signature operator. The Dirac operator $D(\sigma_k)$ is a first-order selfadjoint differential operator with spectrum consisting of real eigenvalues of finite multiplicities $\{\lambda_\ell\}_{\ell \in \mathbb{Z}}$. The eta function $\eta(D(\sigma_k), s)$ is defined by

$$\eta(D(\sigma_k), s) = \sum_{\lambda_\ell > 0} \lambda_\ell^{-s} - \sum_{\lambda_\ell < 0} (-\lambda_\ell)^{-s} \quad \text{for } \mathrm{Re}(s) \gg 0, \tag{2-19}$$

which has a meromorphic extension to $\mathbb{C}$ and is regular at $s = 0$. The eta invariant of $D(\sigma_k)$ is defined by

$$\eta(D(\sigma_k)) = \eta(D(\sigma_k), 0). \tag{2-20}$$

We refer to [2; 9] for more details on the eta invariant.

As explained in [37, Section 3], the following 3–form is a left-invariant form on $\mathrm{PSL}(2,\mathbb{C})$, which can be identified with the frame bundle $F(\mathbb{H}^3)$:

$$C = \frac{1}{4\pi^2}(4\theta_1\wedge\theta_2\wedge\theta_3 - d(\theta_1\wedge\theta_{23}+\theta_2\wedge\theta_{31}+\theta_3\wedge\theta_{12})) + \frac{i}{4\pi^2}(\theta_{12}\wedge\theta_{13}\wedge\theta_{23}-\theta_{12}\wedge\theta_1\wedge\theta_2-\theta_{13}\wedge\theta_1\wedge\theta_3-\theta_{23}\wedge\theta_2\wedge\theta_3). \tag{2-21}$$

Here θ_i and θ_{ij} denote the fundamental form and the connection form, respectively, of the Levi-Civita connection on $F(\mathbb{H}^3)$. The differential of the developing map from the universal cover $\widetilde{\mathcal{M}}_\Gamma$ to $\mathbb{H}^3$ defines a map from $F(\widetilde{\mathcal{M}}_\Gamma)$ to $F(\mathbb{H}^3)$. The pullback of C by this map also descends to the frame bundle $F(\mathcal{M}_\Gamma)$ since it is left-invariant under the action of Γ. Now the complex volume of $\mathcal{M}_\Gamma$ is defined by

$$\frac{1}{\pi^2}\mathbb{V}(\mathcal{M}_\Gamma) := \frac{1}{\pi^2}(\mathrm{Vol}(\mathcal{M}_\Gamma)+i2\pi^2\mathrm{CS}(\mathcal{M}_\Gamma)) = \int_{\mathcal{M}_\Gamma} s^*C \mod i\mathbb{Z}. \tag{2-22}$$

Here s denotes a section from $\mathcal{M}_\Gamma$ to $F(\mathcal{M}_\Gamma)$ and the ambiguity in the phase part of (2-22) by $i\mathbb{Z}$ is due to a choice of s. The Chern–Simons invariant has the following relation to the eta invariant of the odd signature operator $D(\sigma_2)$ over $\mathcal{M}_\Gamma$:

$$2\,\mathrm{CS}(\mathcal{M}_\Gamma) = 3\eta(D(\sigma_2)) \mod \mathbb{Z}. \tag{2-23}$$

Actually this equality holds for any closed Riemannian 3–manifold $\mathcal{M}$. We refer to [3] for more details about this formula.

2.4 Hodge Laplacian

We begin with the general case of a Riemannian manifold. Let $\mathcal{M}$ be an oriented Riemannian manifold of dimension n. For the differential $d\colon \Omega^{p-1}(\mathcal{M}) \to \Omega^p(\mathcal{M})$, its formal adjoint operator $d^*\colon \Omega^p(\mathcal{M}) \to \Omega^{p-1}(\mathcal{M})$ is defined by

$$d^* = (-1)^{np+n+1}\star d\star, \tag{2-24}$$

where $\star\colon \Omega^p(\mathcal{M}) \to \Omega^{n-p}(\mathcal{M})$ is the Hodge star operator with $\star^2 = (-1)^{p(n-p)}\,\mathrm{Id}$ on $\Omega^p(\mathcal{M})$. Then the Hodge Laplacian on $\Omega^p(\mathcal{M})$ is defined by $\Delta_p = (d+d^*)^2$.

For a flat vector bundle E over $\mathcal{M}$, the above operators are extended as follows. Let U be an open subset in $\mathcal{M}$, where $\bigwedge^{p-1}T^*\mathcal{M}$ and E are trivial over U. Let $e_1,\dots,e_r$ be a basis of flat sections of $E|_U$, where r is the rank of E. Then any $\phi \in \Omega^{p-1}(U,E)$ can be written as

$$\phi = \sum_{i=1}^r \phi_i \otimes e_i, \tag{2-25}$$

where $\phi_i \in \Omega^{p-1}(U)$. Now $d\colon \Omega^{p-1}(U,E) \to \Omega^p(U,E)$ is defined by

$$d\phi = \sum_{i=1}^{r} d\phi_i \otimes e_i. \tag{2-26}$$

Note that this is well defined since the flat vector bundle E has a constant transition map. The operator in (2-24) can be extended to an operator $d^{*,\flat}\colon \Omega^p(\mathcal{M},E) \to \Omega^{p-1}(\mathcal{M},E)$ by

$$d^{*,\flat} = (-1)^{np+n+1}(\star \otimes \mathrm{Id}_E)\, d(\star \otimes \mathrm{Id}_E). \tag{2-27}$$

Here note that $(d^{*,\flat})^2 = 0$. Now an extension of the Hodge Laplacian Δ_p on $\Omega^p(\mathcal{M},E)$ is defined by

$$\Delta_p^\flat = (d + d^{*,\flat})^2, \tag{2-28}$$

where d and $d^{*,\flat}$ are as defined in (2-26) and (2-27).

Assuming a Hermitian metric $\langle \cdot,\cdot \rangle_E$ on E, we define the usual formal adjoint operator $d^*\colon \Omega^p(\mathcal{M},E) \to \Omega^{p-1}(\mathcal{M},E)$ extending the operator in (2-24) by

$$d^* = (-1)^{np+n+1}(\star \otimes \mathrm{Id}_E)\mu^{-1}\, d\mu(\star \otimes \mathrm{Id}_E). \tag{2-29}$$

Here $\mu\colon E \to E^*$ is the map defined by

$$\langle u, v\rangle_E = (u, \mu(v)) \tag{2-30}$$

where $(\cdot,\cdot)$ is the dual pairing. We refer to [15, Section 2; 7, Section 8] for more details of this construction. Now the usual Hodge Laplacian on $\Omega^p(\mathcal{M},E)$ is defined by

$$\Delta_p = (d + d^*)^2, \tag{2-31}$$

where d and d^* are as defined in (2-26) and (2-29).

By the definition in (2-29), $\Delta_p^\flat = \Delta_p$ when E is unitarily flat, and for a nonunitary flat vector bundle E, the difference $d^{*,\flat} - d^*$ is a zeroth-order operator. Hence, in general $\Delta_p^\flat - \Delta_p$ is a first-order differential operator on $\Omega^p(\mathcal{M},E)$. For a Hermitian metric on E, we can consider a L^2–completion of $\Omega^p(\mathcal{M},E)$, which is denoted by $L^2(\Omega^p(\mathcal{M},E))$.

Proposition 2.1 *The spectrum of nonselfadjoint operator $\Delta_p^\flat$ on $L^2(\Omega^p(\mathcal{M},E))$ is discrete and consists of generalized eigenvalues of finite multiplicities, which are* *contained in the set* $B_r \cup \lambda_\epsilon$ *for some* $r > 0$ *and* $\epsilon > 0$*, where* $B_r = \{z \in \mathbb{C} : |z| < r\}$ *and* $\Lambda_\epsilon = \{re^{i\theta} \in \mathbb{C} : |\theta| \le \epsilon\}$.

Proof This follows from Theorems 8.4 and 9.3 of [31]. □

The above general construction applies to the case of the nonunitary vector bundle E_ρ over a hyperbolic 3–manifold $\mathcal{M}_\Gamma$, where $\rho = \rho^m$ is the representation of G given in (2-11). In particular, we have the operator $\Delta_p^\flat$ acting on $\Omega^p(\mathcal{M}_\Gamma, E_\rho)$. Let us denote by $\bar{\Delta}_p^\flat$ the lifting of $\Delta_p^\flat$ on the universal covering space $\widetilde{\mathcal{M}}_\Gamma \cong G/K$, and let $\widetilde{E}_\rho = p^* E_\rho$ for the natural projection $p\colon \widetilde{\mathcal{M}}_\Gamma \to \mathcal{M}_\Gamma$. Then we have

$$\Omega^p(\widetilde{\mathcal{M}}_\Gamma, \widetilde{E}_\rho) \cong \big(C^\infty(G) \otimes \textstyle\bigwedge^p \mathfrak{p}^*\big)^K \otimes \widetilde{E}_\rho, \tag{2-32}$$

where $k \in K$ acts by $R(k) \otimes \bigwedge^p \mathrm{Ad}_{\mathfrak{p}^*}(k)$, and the vector bundle $\bigwedge^p T^*\mathcal{M}_\Gamma$ is given as a locally homogeneous vector bundle E_τ for the representation $\tau = \bigwedge^p \mathrm{Ad}_{\mathfrak{p}^*}$ of K. With respect to (2-32), by Kuga's lemma,

$$\bar{\Delta}_p^\flat = -R(\Omega) \otimes \mathrm{Id}_{V_\rho}. \tag{2-33}$$

2.5 Wave kernel

Throughout this section, for simplicity we denote by L the operator $\Delta_p^\flat$ acting on $\Omega^p(\mathcal{M}_\Gamma, E_\rho) = C^\infty(\mathcal{M}_\Gamma, E_\tau \otimes E_\rho)$, where E_τ and E_ρ are defined as in Section 2.2 with $\tau = \bigwedge^p \mathrm{Ad}_{\mathfrak{p}^*}$ and $\rho = \rho^m$. We follow [22] for the construction of the wave kernel of L, which is a crucial ingredient in the derivation of the Selberg trace formula in our setting.

First we assume that L has no zero generalized eigenvalue, that is, $0 \notin \sigma(L)$. Under this condition, there exists an Agmon angle for L and we can define $L^{1/2}$ following [31, Section 10]. One can prove that the spectrum of $L^{1/2}$ lies in the subset of $\mathbb{C}$ with conditions $\mathrm{Re}(\lambda) > \delta$ and $|\mathrm{Im}(\lambda)| < a$ for some $\delta > 0$ and $a > 0$.

Let $\mathcal{P}(\mathbb{C})$ be the space of Paley–Wiener functions on $\mathbb{C}$. Recall

$$\mathcal{P}(\mathbb{C}) = \bigcup_{R>0} \mathcal{P}^R(\mathbb{C})$$

with the inductive limit topology, where $\mathcal{P}^R(\mathbb{C})$ is the space of entire functions ϕ on $\mathbb{C}$ with the condition: for every $N \in \mathbb{N}$ there exist a $C_N > 0$ such that

$$|\phi(\lambda)| \le C_N (1 + |\lambda|)^{-N} e^{R|\mathrm{Im}(\lambda)|} \quad \text{for } \lambda \in \mathbb{C}. \tag{2-34}$$

Given $h \in C_0^\infty((-R,R))$, let

$$\varphi(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} h(r)e^{-ir\lambda}\,dr \quad \text{for } \lambda\in\mathbb{C}$$

be the Fourier–Laplace transform of h. Then φ satisfies (2-34) for every $N\in\mathbb{N}$, that is, $\varphi\in\mathcal{P}^R(\mathbb{C})$. Conversely, by the Paley–Wiener theorem (see Theorem 7.3.1 of [13]), every $\phi\in\mathcal{P}^R(\mathbb{C})$ is the Fourier–Laplace transform of a function in $C_c^\infty((-R,R))$.

For an even Paley–Wiener function $\varphi\in\mathcal{P}(\mathbb{C})$, we define $\varphi(L^{1/2})$ by

$$\varphi(L^{1/2}):=\frac{i}{2\pi}\int_\Gamma \varphi(\lambda)(L^{1/2}-\lambda\,\mathrm{Id})^{-1}\,d\lambda. \tag{2-35}$$

Here Γ is a counterclockwise oriented smooth curve given by $\Gamma_1\cup\Gamma_2\cup\Gamma_3$, where

$$\Gamma_1=\{\lambda\in\mathbb{C}\mid \mathrm{Im}(\lambda)=a,\ \infty>\mathrm{Re}(\lambda)\ge\delta_1\},$$
$$\Gamma_3=\{\lambda\in\mathbb{C}\mid \mathrm{Im}(\lambda)=-a,\ \delta_1\le\mathrm{Re}(\lambda)<\infty\}$$

for some $\delta>\delta_1>0$, and $\Gamma_2\subset\{\lambda\in\mathbb{C}\mid\delta_1\le\mathrm{Re}(\lambda)<\delta\}$ is a simple curve connecting the finite boundary points of Γ_1 and Γ_3.

For $f\in C^\infty(\mathcal{M}_\Gamma,E_\tau\otimes E_\rho)$, we can express $\varphi(L^{1/2})f$ in terms of the solution of the wave equation

$$\left(\frac{\partial^2}{\partial t^2}+L\right)u=0,\qquad u(0,x)=f(x),\quad u_t(0,x)=0. \tag{2-36}$$

By the construction in [33, Sections IV-1 and IV-2], there exists a unique solution $u(t,f)\in C^\infty(\mathbb{R}\times\mathcal{M}_\Gamma,E_\tau\otimes E_\rho)$ of (2-36). Now, by Proposition 3.2 of [22], for $f\in C^\infty(\mathcal{M}_\Gamma,E_\tau\otimes E_\rho)$,

$$\varphi(L^{1/2})f=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\widehat{\varphi}(t)u(t,f)\,dt, \tag{2-37}$$

where $\widehat{\varphi}$ is the Fourier transform of $\varphi|_{\mathbb{R}}$, that is, $\widehat{\varphi}(t)=(1/\sqrt{2\pi})\int_{\mathbb{R}}\varphi(\lambda)e^{-i\lambda t}\,d\lambda$.

The above construction can be generalized when L has a zero generalized eigenvalue. For this, following [22, Section 2], we put

$$\widehat{L}=L(\mathrm{Id}-\Pi_0)\oplus\Pi_0, \tag{2-38}$$

where Π_0 denotes the orthogonal projection onto the generalized eigenspace V_0 such that there exists an integer N_0 with $L^{N_0}V_0=0$. Since Π_0 is a smoothing operator, $\widehat{L}$ is a pseudodifferential operator with the same symbol as L. Moreover, $\sigma(\widehat{L})$ also lies in the same set $B_r\cup\lambda_\epsilon\subset\mathbb{C}$ and $0\notin\sigma(\widehat{L})$, and $\widehat{L}$ has an Agmon angle. Hence, we can define $\widehat{L}^{1/2}$ as in [31, Section 10]. Let V_1 be the complementary subspace of V_0 which is invariant under L. We can also repeat the construction given in (2-35) for $\widehat{L}_1^{1/2} := \widehat{L}^{1/2}|_{V_1}$ to define $\varphi(\widehat{L}_1^{1/2})$. To deal with the remaining part of L, putting $N := L\Pi_0$ we define

$$\varphi(N^{1/2}) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{\varphi}(t) U(t, N)\, dt, \tag{2-39}$$

where $U(t,N) := \sum_{k=0}^{d} ((-1)^k t^{2k}/(2k)!) N^k$ with $d = \dim(V_0)$. Then, combining these constructions, we define

$$\varphi(L^{1/2}) := \varphi(\widehat{L}_1^{1/2})(\mathrm{Id} - \Pi_0) + \varphi(N^{1/2})\Pi_0. \tag{2-40}$$

By Proposition 3.2 of [22], the expression as in (2-37) holds even when $0 \in \sigma(L)$.

Proposition 2.2 *For an even Paley–Wiener function $\varphi \in \mathcal{P}(\mathbb{C})$ and the operator $L = \Delta_p^\flat$ acting on $\Omega^p(\mathcal{M}_\Gamma, E_\rho)$, the operator $\varphi(L^{1/2})$ is of trace class with the smooth kernel*

$$K_\varphi(\Gamma g_1 K, \Gamma g_2 K) = \sum_{\gamma \in \Gamma} H_\varphi(g_1^{-1}\gamma g_2) \otimes \rho(\gamma), \tag{2-41}$$

where $\Gamma g_1 K, \Gamma g_2 K \in \mathcal{M}_\Gamma \cong \Gamma\backslash G/K$ and $H_\varphi\colon G \to \mathrm{End}(V_\tau)$ is a C^∞–function which satisfies $H_\varphi(k_1 g k_2) = \tau(k_1) \circ H_\varphi(g) \circ \tau(k_2)$ for $k_1, k_2 \in K$.

Proof First, one can show that $\varphi(L^{1/2})$ is of trace class with a smooth kernel K_φ by a standard argument as in Lemma 2.4 of [22]. To derive the expression in (2-41), we consider the liftings of $u(t,x,f)$ and f satisfying (2-36) to $\widetilde{\mathcal{M}}_\Gamma$, which we denote by $\widetilde{u}(t,\widetilde{x},f)$ and $\widetilde{f}$, respectively. Then, for the operator $\widetilde{L}^\flat = \widetilde{\Delta}_p^\flat$ over $\widetilde{\mathcal{M}}_\Gamma$, the lifted solution $\widetilde{u}(t,f)$ satisfies

$$\left(\frac{\partial^2}{\partial t^2} + \widetilde{L}^\flat\right)\widetilde{u}(t,f) = 0, \qquad \widetilde{u}(0,f) = \widetilde{f}, \quad \widetilde{u}_t(0,f) = 0. \tag{2-42}$$

By the energy estimate given in [33, Chapter 2], for every $\psi \in \Omega^p(\widetilde{\mathcal{M}}_\Gamma, \widetilde{E}_\rho)$, the wave equation

$$\left(\frac{\partial^2}{\partial t^2} + \widetilde{L}^\flat\right)u(t,\psi) = 0, \qquad u(0,\psi) = \psi, \quad u_t(0,\psi) = 0 \tag{2-43}$$

has a unique solution. Hence,

$$\widetilde{u}(t,f) = u(t,\widetilde{f}). \tag{2-44}$$

Since $\widetilde{\mathcal{M}}_\Gamma \cong G/K$ with $G = \mathrm{SL}(2,\mathbb{C})$ and $K = \mathrm{SU}(2)$ and

$$\Omega^p(\widetilde{\mathcal{M}}_\Gamma, \widetilde{E}_\rho) \cong C^\infty(G/K, \widetilde{E}_\tau \otimes \widetilde{E}_\rho) \quad \text{for } \tau = \textstyle\bigwedge^p \mathrm{Ad}_{\mathfrak{p}^*},$$

we apply some harmonic analysis for (G,K) to obtain a more explicit expression of (2-44). First, recall that $[\pi|_K : \tau] \le 1$ for any $\pi \in \widehat{G}$ and $\tau \in \widehat{K}$. For $\pi \in \widehat{G}(\tau) = \{\pi \in \widehat{G} \mid [\pi|_K : \tau] = 1\}$, the τ–isotypical subspace $\mathcal{H}_\pi(\tau)$ of τ in $\mathcal{H}_\pi$ can be identified with V_τ. Define a τ–spherical function Φ^π_τ on G by

$$\Phi^\pi_\tau(g) = P_\tau \pi(g) P_\tau \tag{2-45}$$

for $g \in G$, where P_τ denotes the orthogonal projection of $\mathcal{H}_\pi$ onto the τ–isotypical subspace $\mathcal{H}_\pi(\tau)$. Moreover, we have the identification

$$C^\infty(G/K, \widetilde{E}_\tau) \cong C^\infty(G;\tau), \tag{2-46}$$

where

$$C^\infty(G;\tau) = \{ f \in C^\infty(G, V_\tau) \mid f(gk) = \tau(k^{-1}) f(g),\ g \in G,\ k \in K\}. \tag{2-47}$$

Then, for $f^\pi_{\tau,v} := \Phi^\pi_\tau(g^{-1})(v) \in C^\infty(G;\tau)$ with $v \in V_\tau$, we define

$$\widetilde{L} f^\pi_{\tau,v} = -\pi(\Omega) f^\pi_{\tau,v}. \tag{2-48}$$

Hence, the unique solution $u(t,x,f^\pi_{\tau,v})$ with $u(0) = f^\pi_{\tau,v}$ of the corresponding equation with $\widetilde{L}$ to (2-43) is given by

$$u(t,x,f^\pi_{\tau,v}) = \cos(t\sqrt{-\pi(\Omega)})\, f^\pi_{\tau,v}. \tag{2-49}$$

This immediately implies

$$\begin{aligned} \varphi(\widetilde{L}^{1/2}) f^\pi_{\tau,v} &= \varphi(\sqrt{-\pi(\Omega)})\, f^\pi_{\tau,v} \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\varphi}(t) \cos(t\sqrt{-\pi(\Omega)})\, f^\pi_{\tau,v}\, dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\varphi}(t) u(t,x,f^\pi_{\tau,v})\, dt. \end{aligned} \tag{2-50}$$

In the same way as Proposition 3.3 of [22], one can show that $\varphi(\widetilde{L}^{1/2})$ has a smooth kernel $\widetilde{K}_\varphi \in C^\infty(\widetilde{\mathcal{M}}_\Gamma \times \widetilde{\mathcal{M}}_\Gamma, \mathrm{Hom}(\widetilde{E}_\tau, \widetilde{E}_\tau))$ such that for $\psi \in C^\infty(\widetilde{\mathcal{M}}_\Gamma, \widetilde{E}_\tau)$,

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\varphi}(t) u(t,\tilde{x},\psi)\, dt = \int_{\widetilde{\mathcal{M}}_\Gamma} \widetilde{K}_\varphi(\tilde{x},\tilde{y}) \psi(\tilde{y})\, d\tilde{y}. \tag{2-51}$$

Since $\varphi(\widetilde{L}^{1/2})$ is a G–invariant integral operator, its kernel $\widetilde{K}_\varphi$ satisfies

$$\widetilde{K}_\varphi(g\tilde{x}, g\tilde{y}) = \widetilde{K}_\varphi(\tilde{x}, \tilde{y}) \quad \text{for } g \in G. \tag{2-52}$$

With respect to (2-46), the kernel $\widetilde{K}_\varphi$ can be identified with a C^∞–function $H_\varphi\colon G \to \mathrm{End}(V_\tau)$, which satisfies

$$H_\varphi(k_1 g k_2) = \tau(k_1) \circ H_\varphi(g) \circ \tau(k_2) \quad \text{for } k_1, k_2 \in K. \tag{2-53}$$

Then $\varphi(\widetilde{L}^{1/2})$ acts by convolution:

$$(\varphi(\widetilde{L}^{1/2}) f)(g_1) = \int_G H_\varphi(g_1^{-1} g_2) f(g_2)\, dg_2. \tag{2-54}$$

As in the proof of Proposition 1.21 of [6], combining (2-37), (2-44) and (2-51), the following equality holds for $f \in C^\infty(\mathcal{M}_\Gamma, E_\tau \otimes E_\rho)$:

$$\varphi(L^{1/2}) f(\tilde{x}) = \int_{\widetilde{\mathcal{M}}_\Gamma} (\widetilde{K}_\varphi(\tilde{x}, \tilde{y}) \otimes \mathrm{Id}_{V_\rho}) f(\tilde{y})\, d\tilde{y}.$$

For a fundamental domain F in $\widetilde{\mathcal{M}}_\Gamma$ of the action of Γ and the induced bundle map $R_\gamma\colon \widetilde{E}_{\tilde{y}} \to \widetilde{E}_{\gamma\tilde{y}}$,

$$\begin{aligned}
\int_{\widetilde{\mathcal{M}}_\Gamma} (\widetilde{K}_\varphi(\tilde{x}, \tilde{y}) \otimes \mathrm{Id}_{V_\rho}) f(\tilde{y})\, d\tilde{y} &= \sum_{\gamma\in\Gamma} \int_{\gamma F} (\widetilde{K}_\varphi(\tilde{x}, \tilde{y}) \otimes \mathrm{Id}_{V_\rho}) f(\tilde{y})\, d\tilde{y} \\
&= \sum_{\gamma\in\Gamma} \int_{F} (\widetilde{K}_\varphi(\tilde{x}, \gamma\tilde{y}) \otimes \mathrm{Id}_{V_\rho}) f(\gamma\tilde{y})\, d\tilde{y} \\
&= \int_F \Bigl(\sum_{\gamma\in\Gamma} \widetilde{K}_\varphi(\tilde{x}, \gamma\tilde{y}) \circ (R_\gamma \otimes \rho(\gamma)) \Bigr) f(\tilde{y})\, d\tilde{y}.
\end{aligned}$$

With respect to (2-46), for $\tilde{x} = g_1 K$ and $\tilde{y} = g_2 K$,

$$\sum_{\gamma\in\Gamma} \widetilde{K}_\varphi(\tilde{x}, \gamma\tilde{y}) \circ (R_\gamma \otimes \rho(\gamma)) = \sum_{\gamma\in\Gamma} H_\varphi(g_1^{-1}\gamma g_2) \otimes \rho(\gamma).$$

From this, one can see that the kernel K_φ of $\varphi(L^{1/2})$ has the form given in (2-41), which also satisfies the claimed property by (2-53). This completes the proof. □

By Proposition 2.2, applying Lidskii's theorem (see Theorem 8.4 of [10]), we have:

Corollary 2.3 *For an even Paley–Wiener function $\varphi \in \mathcal{P}(\mathbb{C})$ and the operator $L = \Delta_p^\flat$ acting on $\Omega^p(\mathcal{M}_\Gamma, E_\rho)$,*

$$\sum_{\lambda \in \mathrm{spec}(L)} m(\lambda)\varphi(L^{1/2}) = \mathrm{Tr}(\varphi(L^{1/2})) \tag{2-55}$$
$$= \int_{\mathcal{M}_\Gamma} \mathrm{tr}\, K_\varphi(m,m)\, dm$$
$$= \sum_{\gamma\in\Gamma} \mathrm{tr}\,\rho(\gamma) \int_{\Gamma\backslash G} h_\varphi(g^{-1}\gamma g)\, d\dot{g},$$

where $m(\lambda)$ is the multiplicity of the generalized eigenvalue λ of L and $h_\varphi = \mathrm{tr}\, H_\varphi$.

2.6 Selberg trace formula

Proposition 2.4 *For the operator $\Delta_p^\flat$ acting on $\Omega^p(\mathcal{M}_\Gamma, E_\rho)$,*

$$\mathrm{Tr}(e^{-t(\Delta_0^\flat - 1)}) = \dim(V_\rho)\,\mathrm{Vol}(\mathcal{M}_\Gamma)\frac{1}{4\pi^2}\int_{\mathbb{R}} \lambda^2 e^{-t\lambda^2}\, d\lambda + \sum_{[\gamma]\neq[e]} \frac{\ell_{\gamma_0}\,\mathrm{tr}\,\rho(\gamma)}{D(\gamma)} \frac{1}{\sqrt{4\pi t}} e^{-\ell_\gamma^2/4t} \tag{2-56}$$

and

$$\mathrm{Tr}(e^{-t\Delta_1^\flat}) - \mathrm{Tr}(e^{-t\Delta_0^\flat}) = \dim(V_\rho)\,\mathrm{Vol}(\mathcal{M}_\Gamma)\frac{1}{2\pi^2}\int_{\mathbb{R}} (\lambda^2+1) e^{-t\lambda^2}\, d\lambda + \sum_{[\gamma]\neq[e]} \frac{\ell_{\gamma_0}\,\mathrm{tr}\,\rho(\gamma)}{D(\gamma)} (e^{i\theta_\gamma} + e^{-i\theta_\gamma}) \frac{1}{\sqrt{4\pi t}} e^{-\ell_\gamma^2/4t}, \tag{2-57}$$

where $D(\gamma) = e^{\ell_\gamma}|1 - e^{-(\ell_\gamma + i\theta_\gamma)}|^2$ is as defined in (2-10) *and the sums on the right-hand sides run over the set of conjugacy classes in Γ of nontrivial elements in Γ.*

Remark 2.5 The corresponding formulas to (2-56) and (2-57) are well known for the selfadjoint operators Δ_p, which can be derived from Theorem 6.7 of [35] easily. After this paper was completed, it is also known to the author that the equalities (2-56) and (2-57) can be derived from Theorem 5.5 of [32] and Theorem 6.2 of [8].

Proof Since the proofs are essentially the same, we provide only the proof of the equality (2-57), which consists of two parts.

Firstly, we proceed as in the original way of Selberg from the equality (2-55) to analyze the orbital integral terms on right-hand side of (2-55). For more details about this we refer to [28, Section 4]. To state the resulting formula, let $\Theta_{k,\lambda}$ denote the character of the induced representation $(\pi_{k,\lambda}, \mathcal{H}_{k,\lambda})$, where

$$\pi_{k,\lambda} = \mathrm{Ind}_{MAN}^{G}(\sigma_k \otimes e^{i\lambda} \otimes 1). \tag{2-58}$$

Then, for an even Paley–Wiener function $\varphi \in \mathcal{P}(\mathbb{C})$, we have

$$\begin{aligned}\sum_{\lambda \in \mathrm{spec}(\Delta_1^\flat)} m(\lambda)\varphi(\lambda^{1/2}) - \sum_{\lambda \in \mathrm{spec}(\Delta_0^\flat)} m(\lambda)\varphi(\lambda^{1/2}) &= \dim(V_\rho)\,\mathrm{Vol}(\mathcal{M}_\Gamma)\frac{1}{2\pi^2}\int_{\mathbb{R}} \Theta_{2,\lambda}(h_\varphi)(\lambda^2+1)\,d\lambda \\ &\quad + \sum_{[\gamma]\neq[e]} \frac{\ell_{\gamma_0}\,\mathrm{tr}\,\rho(\gamma)}{D(\gamma)}(e^{i\theta_\gamma}+e^{-i\theta_\gamma})\frac{1}{2\pi}\int_{\mathbb{R}} \Theta_{2,\lambda}(h_\varphi)e^{-i\ell_\gamma\lambda}\,d\lambda.\end{aligned} \tag{2-59}$$

Although we do not spell out the details of the derivation of this equality, let us make some remarks. Since we consider the difference of the traces for $\Delta_1^\flat$ and $\Delta_0^\flat$ on the spectral side of (2-59), the terms involving $\Theta_{0,\lambda}(h_\varphi)$ cancel out each other on the geometric side of (2-59). Note that for the first term on the right-hand side of (2-59), we used

$$\mathrm{Vol}(\mathcal{M}_\Gamma) = \pi\,\mathrm{Vol}(\Gamma\backslash G), \tag{2-60}$$

which can be derived in the same way as equation (4.31) of [11]. Let us recall the equality given in (2.12) of [23],

$$\pi_{k,\lambda}(\Omega) = -\lambda^2 + \tfrac{1}{4}k^2 - 1. \tag{2-61}$$

Then, combining this and (2-50),

$$\Theta_{\pm 2,\lambda}(h_\varphi) = \varphi\big(\sqrt{-\pi_{\pm 2,\lambda}(\Omega)}\big) = \varphi(\lambda). \tag{2-62}$$

Here we used that φ is even for the second equality. Hence, we have that $\Theta_{2,\lambda}(h_\varphi) = \Theta_{-2,\lambda}(h_\varphi)$, which was also used to simplify the right-hand side of (2-59).

Secondly, we want to have the equality (2-59) for the test function $\varphi_t(\lambda) = e^{-t\lambda^2}$, which is not given by a Fourier–Laplace transform of a compactly supported function. We follow the proof of Theorem 1.27 in [6] for this. Since our case is very specific, that is, $G = \mathrm{SL}(2,\mathbb{C})$ and $\varphi_t = e^{-t\lambda^2}$, it can be given in an explicit way as follows.

Recall that

$$\check{\varphi}_t(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t\lambda^2} e^{is\lambda}\, d\lambda = \frac{1}{\sqrt{2t}} e^{-s^2/4t}, \tag{2-63}$$

and let $\chi_R\colon \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\chi_R(s) = 1$ for $|s| \geq R - \delta$ and $\chi_R(s) = 0$ for $|s| < R - 1 + \delta$, where $0 < \delta < \frac{1}{2}$ is a constant independent of t. We put

$$\check{\varphi}_{t,R}(s) = \check{\varphi}_t(s) - \chi_R(s)\check{\varphi}_t(s).$$

Now the equality (2-59) is valid for a test function whose inverse Fourier–Laplace transform is $\check{\varphi}_{t,R}$. Then, by (2-62), the resulting equality can be split into terms with $\varphi_t(s)$ and terms with $f_{t,R}$, where

$$f_{t,R}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \chi_R(s)\check{\varphi}_t(s) e^{-is\lambda}\, ds = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \chi_R(s) \frac{1}{\sqrt{2t}} e^{-s^2/4t} e^{-is\lambda}\, ds.$$

By integration by parts, for any $N \in \mathbb{N}$ we have

$$\begin{aligned} f_{t,R}(\lambda) &= \frac{1}{\sqrt{4\pi t}} (i\lambda)^{-N} \int_{A_R} e^{-i\lambda s} \partial_s^N (\chi_R(s) e^{-s^2/4t})\, ds \\ &= \frac{1}{\sqrt{4\pi t}} (i\lambda)^{-N} \int_{A_R} e^{-i\lambda s} \left(e^{-s^2/4t} \sum_{k=0}^{N} \left(\left(-\frac{s}{2t}\right)^k \chi_R^{(N-k)}(s) \right) \right) ds, \end{aligned} \tag{2-64}$$

where $A_R := (-\infty, -R+1] \sqcup [R-1, \infty)$. In order to obtain the equality (2-57), it is enough to show that terms with $f_{t,R}$ vanish as $R \to \infty$. By (2-64), it is easy to see that the integrals on the right-hand side of (2-59) with $\Theta_{2,\lambda}(h_\varphi) = \varphi(\lambda)$ replaced by $f_{t,R}(\lambda)$ vanish as $R \to \infty$. To deal with the corresponding left-hand side of (2-59), let us observe that there exists $N \in \mathbb{N}$ such that

$$\sum_{\lambda \in \operatorname{spec}(\Delta_i^\flat)} m(\lambda) \lambda^{-N} < \infty \tag{2-65}$$

for $i = 0, 1$. This follows from the Weyl law for the operator $\Delta_i^\flat$, which can be proved as in Lemma 2.2 of [22]. Using (2-64), (2-65) and Proposition 2.1, it is easy to see that the terms

$$\sum_{\lambda \in \operatorname{spec}(\Delta_1^\flat)} m(\lambda) f_{t,R}(\lambda^{1/2}) - \sum_{\lambda \in \operatorname{spec}(\Delta_0^\flat)} m(\lambda) f_{t,R}(\lambda^{1/2})$$

on the left-hand side of (2-59) also vanish as $R \to \infty$. This completes the proof of (2-57). □

3 Zeta functions

For a discrete subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$, the *critical exponent* $\delta(\Gamma)$ is defined by

$$\delta(\Gamma) = \inf\Big\{ u \in \mathbb{R} \;\Big|\; \sum_{\gamma \in \Gamma} e^{-u\ell_\gamma} < \infty \Big\}. \tag{3-1}$$

It is known that $\delta(\Gamma) < 2$ for a cofinite torsion-free subgroup Γ (see Theorem 3.14.1 of [14]). By Theorem 6 of [1, page 192], from this fact one can see that, for $s \in \mathbb{C}$,

$$\prod_{[\gamma] \neq [e]} (1 - e^{-s\ell_\gamma}) < \infty \quad \text{if } \mathrm{Re}(s) > 2, \tag{3-2}$$

where the product runs over the set of conjugacy classes in Γ of the nontrivial elements in Γ. First we put

$$R(\sigma_k, s) = \prod_{[\gamma]_\mathrm{p}} (1 - e^{(k/2)i\theta_\gamma} e^{-sl_\gamma}) \quad \text{for } \mathrm{Re}(s) > 2. \tag{3-3}$$

Here and from now on, the product notation with $[\gamma]_\mathrm{p}$ always means that the product runs over the set of conjugacy classes in Γ of the *primitive* loxodromic elements in Γ. It is well known that $R(\sigma_k, s)$ has a meromorphic extension to $\mathbb{C}$ (see [6]).

For an $\mathrm{SL}(N,\mathbb{C})$–representation (χ, V_χ) of Γ, we assume that all the eigenvalues $\lambda_i(\gamma)$ for $i = 1, \dots, N$ of $\chi(\gamma)$ for $\gamma \in \Gamma$ satisfy

$$|\lambda_i(\gamma)| \leq e^{c_\chi l_\gamma} \tag{3-4}$$

for a constant $c_\chi > 0$ which does not depend on γ. Then the *Ruelle zeta function* $R_\chi(s)$ attached to χ is defined by

$$R_\chi(s) = \prod_{[\gamma]_\mathrm{p}} \det(\mathrm{Id}_{V_\chi} - \chi(\gamma) e^{-sl_\gamma}) \quad \text{for } \mathrm{Re}(s) \gg 0. \tag{3-5}$$

We are interested in the case $\chi = \rho|_\Gamma$, where $\rho = \rho^m$ is a representation of G. Now we have:

Proposition 3.1 *For the restriction to Γ of the representation ρ^m of G, the Ruelle zeta function attached to ρ^m has the expression*

$$R_{\rho^m}(s) = \prod_{l=0}^{m} R\big(\sigma_{m-2l}, s - \tfrac{1}{2}m + l\big) \quad \textit{for } \mathrm{Re}(s) > 2 + \tfrac{1}{2}m, \tag{3-6}$$

and $R_{\rho^m}(s)$ has a meromorphic extension to $\mathbb{C}$.

Proof The equality (3-6) follows from (2-8) and (2-12) easily. Then, the meromorphic extension of $R_{\rho^m}(s)$ to $\mathbb{C}$ follows from that of $R(\sigma_k, s)$. $\square$

Let us remark that Proposition 3.1 also follows from the equality (3.14) of [23].

For an $\mathrm{SL}(N, \mathbb{C})$–representation (χ, V_χ) of Γ satisfying (3-4), the *Selberg zeta function* $Z_\chi(\sigma_k, s)$ attached to χ is defined by

$$Z_\chi(\sigma_k, s) = \prod_{[\gamma]_{\mathrm{p}}} \prod_{\substack{p\geq 0\\ q\geq 0}} \det(\mathrm{Id}_{V_\chi} - \chi(\gamma) e^{(k/2)i\theta_\gamma} e^{-p(l_\gamma+i\theta_\gamma)} e^{-q(l_\gamma-i\theta_\gamma)} e^{-sl_\gamma}) \qquad \text{for } \mathrm{Re}(s) \gg 0.$$

For the trivial χ, we denote it by $Z(\sigma_k, s)$, and the convergence of $Z(\sigma_k, s)$ for $\mathrm{Re}(s) > 2$ follows from (3-2) by a similar computation to (4-8). It is also well known that $Z(\sigma_k, s)$ has a meromorphic extension to $\mathbb{C}$ (see [6; 11]). For $\chi = \rho^m$, the Selberg zeta function $Z_{\rho^m}(\sigma_k, s)$ has the following expression by (2-8) and (2-12):

$$Z_{\rho^m}(\sigma_k, s) = \prod_{l=0}^{m} Z\big(\sigma_{m-2l+k}, s - \tfrac{1}{2}m + l\big) \quad \text{for } \mathrm{Re}(s) > 2 + \tfrac{1}{2}m, \tag{3-7}$$

which has a meromorphic extension to $\mathbb{C}$.

In the following two propositions, we will derive two expressions for $R_{\rho^m}(s)$ in terms of the Selberg zeta functions. These two formulas and their relationship are the starting point for the proofs of main results of this paper.

Proposition 3.2 *Over* $\mathbb{C}$,

$$R_{\rho^m}(s) = \frac{Z\big(\sigma_m, s - \frac{1}{2}m\big) Z\big(\sigma_{-m}, s + \frac{1}{2}m + 2\big)}{Z\big(\sigma_{m+2}, s - \frac{1}{2}m + 1\big) Z\big(\sigma_{-(m+2)}, s + \frac{1}{2}m + 1\big)}. \tag{3-8}$$

Proof Since the Selberg zeta function $Z(\sigma_k, s)$ attached to σ_k has a meromorphic extension over $\mathbb{C}$, it is sufficient to show the equality (3-8) over a domain where both sides converge absolutely. Over such a domain, we have

$$\begin{aligned} Z\big(\sigma_m, s - \tfrac{1}{2}m\big) &= \prod_{[\gamma]_{\mathrm{p}}} \prod_{\substack{p\geq 0\\ q\geq 0}} (1 - e^{(m/2)i\theta_\gamma} e^{-p(l_\gamma+i\theta_\gamma)} e^{-q(l_\gamma-i\theta_\gamma)} e^{-(s-m/2)l_\gamma}) \\ &= \prod_{[\gamma]_{\mathrm{p}}} \prod_{\substack{p\geq 0\\ q\geq 0}} (1 - e^{(m/2)(\ell_\gamma+i\theta_\gamma)} e^{-p(l_\gamma+i\theta_\gamma)} e^{-q(l_\gamma-i\theta_\gamma)} e^{-sl_\gamma}). \end{aligned} \tag{3-9}$$

Similarly, we have

$$Z\big(\sigma_{m+2}, s-\tfrac{1}{2}m+1\big) = \prod_{[\gamma]_{\mathrm{p}}} \prod_{\substack{p\geq 0\\ q\geq 0}} (1-e^{(m/2+1)i\theta_\gamma} e^{-p(l_\gamma+i\theta_\gamma)} e^{-q(l_\gamma-i\theta_\gamma)} e^{-(s-m/2+1)l_\gamma}) = \prod_{[\gamma]_{\mathrm{p}}} \prod_{\substack{p\geq 0\\ q\geq 1}} (1-e^{(m/2)(\ell_\gamma+i\theta_\gamma)} e^{-p(l_\gamma+i\theta_\gamma)} e^{-q(l_\gamma-i\theta_\gamma)} e^{-sl_\gamma}). \tag{3-10}$$

By (3-9) and (3-10),

$$\frac{Z\big(\sigma_m, s-\frac{1}{2}m\big)}{Z\big(\sigma_{m+2}, s-\frac{1}{2}m+1\big)} = \prod_{[\gamma]_{\mathrm{p}}} \prod_{p\geq 0} (1-e^{(m/2)(\ell_\gamma+i\theta_\gamma)} e^{-p(l_\gamma+i\theta_\gamma)} e^{-sl_\gamma}). \tag{3-11}$$

In the same way,

$$\frac{Z\big(\sigma_{-m}, s+\frac{1}{2}m+2\big)}{Z\big(\sigma_{-(m+2)}, s+\frac{1}{2}m+1\big)} = \prod_{[\gamma]_{\mathrm{p}}} \prod_{p\geq 1} (1-e^{-(m/2)(\ell_\gamma+i\theta_\gamma)} e^{-p(l_\gamma+i\theta_\gamma)} e^{-sl_\gamma})^{-1}. \tag{3-12}$$

By (3-11) and (3-12),

$$\frac{Z\big(\sigma_m, s-\frac{1}{2}m\big) Z\big(\sigma_{-m}, s+\frac{1}{2}m+2\big)}{Z\big(\sigma_{m+2}, s-\frac{1}{2}m+1\big) Z\big(\sigma_{-(m+2)}, s+\frac{1}{2}m+1\big)} = \prod_{[\gamma]_{\mathrm{p}}} \prod_{k=0}^{m} (1-e^{(m/2-k)(\ell_\gamma+i\theta_\gamma)} e^{-sl_\gamma}) = \prod_{k=0}^{m} R\big(\sigma_{m-2k}, s-\tfrac{1}{2}m+k\big). \tag{3-13}$$

Combining this and Proposition 3.1 completes the proof. □

Proposition 3.3 *Over* $\mathbb{C}$,

$$R_{\rho^m}(s) = \frac{Z_{\rho^m}(\sigma_0, s) Z_{\rho^m}(\sigma_0, s+2)}{Z_{\rho^m}(\sigma_2, s+1) Z_{\rho^m}(\sigma_{-2}, s+1)}. \tag{3-14}$$

Proof By (3-7), we have

$$\frac{Z_{\rho^m}(\sigma_0, s)}{Z_{\rho^m}(\sigma_{-2}, s+1)} = \frac{Z\big(\sigma_m, s-\frac{1}{2}m\big)}{Z\big(\sigma_{-(m+2)}, s+\frac{1}{2}m+1\big)}, \tag{3-15}$$

and

$$Z_{\rho^m}(\sigma_0, s+2) / Z_{\rho^m}(\sigma_2, s+1)$$

(3-16)
$$\frac{Z_{\rho^m}(\sigma_0, s+2)}{Z_{\rho^m}(\sigma_2, s+1)} = \frac{Z\bigl(\sigma_{-m}, s+\frac{1}{2}m+2\bigr)}{Z\bigl(\sigma_{m+2}, s-\frac{1}{2}m+1\bigr)}.$$

Combining (3-15), (3-16) and (3-8) completes the proof. □

Now let us introduce one of the main ingredients of this paper. This is defined by the infinite products of the Ruelle zeta functions attached to σ_{-m} for even or odd integers $m \in \mathbb{N}$:

(3-17)
$$F_n(s) = \prod_{k=n}^{\infty} R(\sigma_{-2k}, s+k) \qquad \text{for } \operatorname{Re}(s) > 2-n,\ n \in \mathbb{N},$$

(3-18)
$$G_n(s) = \prod_{k=n}^{\infty} R\bigl(\sigma_{-(2k+1)}, s+k+\tfrac{1}{2}\bigr) \quad \text{for } \operatorname{Re}(s) > \tfrac{3}{2}-n,\ n \in \mathbb{N}\cup\{0\}.$$

The convergence of $F_n(s)$ and $G_n(s)$ over each half-plane follows from (3-2). Note that a spin structure is involved in the construction of $G_n(s)$. The relation of $F_n(s)$ and $G_n(s)$ with other zeta functions is given as follows:

Proposition 3.4 *The functions $F_n(s)$ and $G_n(s)$ have meromorphic extensions to $\mathbb{C}$ and they satisfy the following relations with Selberg zeta functions*:

(3-19)
$$F_n(s) = \frac{Z(\sigma_{-2n}, s+n)}{Z(\sigma_{-2(n-1)}, s+n+1)},$$

(3-20)
$$G_n(s) = \frac{Z\bigl(\sigma_{-(2n+1)}, s+n+\frac{1}{2}\bigr)}{Z\bigl(\sigma_{-(2n-1)}, s+n+\frac{3}{2}\bigr)}.$$

Proof The equality of (3-19) over the absolute convergence domain $\operatorname{Re}(s) > 2-n$ implies the meromorphic extension of $F_n(s)$ to $\mathbb{C}$. Hence it suffices to show the equality over the convergence domain. Over the absolute convergence domain, we have

(3-21)
$$\begin{aligned} F_n(s) &= \prod_{k=n}^{\infty} \prod_{[\gamma]_{\mathrm{p}}} (1 - e^{-k(\ell_\gamma + i\theta_\gamma)} e^{-s\ell_\gamma}) \\ &= \prod_{[\gamma]_{\mathrm{p}}} \prod_{\substack{p\geq 0\\ q\geq 0}} \frac{1 - e^{-(p+n)(\ell_\gamma+i\theta_\gamma)} e^{-q(\ell_\gamma - i\theta_\gamma)} e^{-s\ell_\gamma}}{1 - e^{-(p+n)(\ell_\gamma+i\theta_\gamma)} e^{-(q+1)(\ell_\gamma - i\theta_\gamma)} e^{-s\ell_\gamma}} \\ &= \frac{Z(\sigma_{-2n}, s+n)}{Z(\sigma_{-2(n-1)}, s+n+1)}. \end{aligned}$$

As above, the meromorphic extension of $G_n(s)$ can be proved by the equality, over the absolute convergence domain $\mathrm{Re}(s) > \frac{3}{2} - n$,

$$G_n(s) = \prod_{k=n}^{\infty} \prod_{[\gamma]_{\mathrm{p}}} (1 - e^{-(k+1/2)(\ell_\gamma + i\theta_\gamma)} e^{-s\ell_\gamma}) \tag{3-22}$$
$$= \prod_{[\gamma]_{\mathrm{p}}} \prod_{\substack{p \geq 0 \\ q \geq 0}} \frac{1 - e^{-(p+n+1/2)(\ell_\gamma + i\theta_\gamma)} e^{-q(\ell_\gamma - i\theta_\gamma)} e^{-s\ell_\gamma}}{1 - e^{-(p+n+1/2)(\ell_\gamma + i\theta_\gamma)} e^{-(q+1)(\ell_\gamma - i\theta_\gamma)} e^{-s\ell_\gamma}}$$
$$= \frac{Z\big(\sigma_{-(2n+1)}, s+n+\frac{1}{2}\big)}{Z\big(\sigma_{-(2n-1)}, s+n+\frac{3}{2}\big)}.$$

This concludes the proof. □

By Propositions 3.2 and 3.4, we have:

Corollary 3.5 *For $s \in \mathbb{C}$ and $n \in \mathbb{N}$,*

$$F_n(s)^2 R_{\rho^{2(n-1)}}(s) = \frac{Z(\sigma_{2(n-1)}, s-n+1) Z(\sigma_{-2n}, s+n)}{Z(\sigma_{-2(n-1)}, s+n+1) Z(\sigma_{2n}, s-n+2)}, \tag{3-23}$$

$$G_n(s)^2 R_{\rho^{2n-1}}(s) = \frac{Z\big(\sigma_{2n-1}, s-n+\frac{1}{2}\big) Z\big(\sigma_{-(2n+1)}, s+n+\frac{1}{2}\big)}{Z\big(\sigma_{2n+1}, s-n+\frac{3}{2}\big) Z\big(\sigma_{-(2n-1)}, s+n+\frac{3}{2}\big)}. \tag{3-24}$$

The following theorem can be considered as a primitive form of Theorems 5.2 and 5.3:

Theorem 3.6 *The function $F_n(s)^2 R_{\rho^{2(n-1)}}(s)$ is regular at $s = 0$ and*

$$F_n(s)^4 R_{\rho^{2(n-1)}}(s)^2|_{s=0} = \exp\Big(-\frac{2}{\pi}\big(2n^2 - 2n + \tfrac{1}{3}\big) \mathrm{Vol}(\mathcal{M}_\Gamma) - 2\pi i \big(\eta(D(\sigma_{2n})) - \eta(D(\sigma_{2(n-1)}))\big)\Big). \tag{3-25}$$

The function $G_n(s)^2 R_{\rho_{2n-1}}(s)$ is regular at $s = 0$ and

$$G_n(s)^4 R_{\rho^{2n-1}}(s)^2|_{s=0} = \exp\Big(-\frac{2}{\pi}\big(2n^2 - \tfrac{1}{6}\big) \mathrm{Vol}(\mathcal{M}_\Gamma) - 2\pi i \big(\eta(D(\sigma_{2n+1})) - \eta(D(\sigma_{2n-1}))\big)\Big). \tag{3-26}$$

Remark 3.7 By the proof of Proposition 3.4, in particular, the first equality in (3-21), $F_n(s)$ has a finite value at $s = 0$ for $n \geq 3$. Similarly, by the first equality in (3-22), $G_n(s)$ has a finite value at $s = 0$ for $n \geq 2$. Hence, from these facts and equalities (3-25) and (3-26), one can see that $R_{\rho^m}(s)$ is regular and has a finite nonzero value at $s = 0$ for $m \geq 3$.

Proof We prove only the case $F_n(s)$ since the proof for $G_n(s)$ is the same. We start by rewriting (3-23) as

$$F_n(s)^2 R_{\rho^{2(n-1)}}(s) = \frac{Z(\sigma_{2(n-1)}, s-n+1)}{Z(\sigma_{-2(n-1)}, -s+n+1)} \frac{Z(\sigma_{-2(n-1)}, -s+n+1)}{Z(\sigma_{-2(n-1)}, s+n+1)} \cdot \frac{Z(\sigma_{-2n}, -s+n)}{Z(\sigma_{2n}, s-n+2)} \frac{Z(\sigma_{-2n}, s+n)}{Z(\sigma_{-2n}, -s+n)}. \tag{3-27}$$

By Theorem 3.18 of [6],

$$Z(\sigma_k, 1+s) = e^{i\pi\eta(D(\sigma_k))} \exp\Bigl(\frac{\mathrm{Vol}(\mathcal{M}_\Gamma)}{\pi}\bigl(\tfrac{1}{3}s^3 - \tfrac{1}{4}k^2 s\bigr)\Bigr) Z(\sigma_{-k}, 1-s). \tag{3-28}$$

By (3-27) and (3-28),

$$\begin{aligned} F_n(s)^2 R_{\rho^{2(n-1)}}(s)|_{s=0} &= \exp\Bigl(-\frac{1}{\pi}\bigl(2n^2-2n+\tfrac{1}{3}\bigr)\mathrm{Vol}(\mathcal{M}_\Gamma) - i\pi\bigl(\eta(D(\sigma_{2n})) - \eta(D(\sigma_{2(n-1)}))\bigr)\Bigr) \\ &\quad \cdot \Bigl(\frac{Z(\sigma_{-2(n-1)}, -s+n+1)}{Z(\sigma_{-2(n-1)}, s+n+1)} \frac{Z(\sigma_{-2n}, s+n)}{Z(\sigma_{-2n}, -s+n)}\Bigr)\Bigr|_{s=0}. \end{aligned} \tag{3-29}$$

The last part evaluated at $s=0$ on the right-hand side of (3-29) need not be 1 since each factor given by the Selberg zeta function may have a zero or a pole at $s=0$. Hence, the concerning part is 1 or -1 in general. We remove this ambiguity by taking the square to obtain (3-25). □

Theorem 3.8 *The following equalities hold*:

$$F_1(s)^2 R_{\rho^0}(s)|_{s=0} = -\exp\Bigl(-\frac{1}{3\pi}\bigl(\mathrm{Vol}(\mathcal{M}_\Gamma) + i3\pi^2\eta(D(\sigma_2))\bigr)\Bigr), \tag{3-30}$$

$$G_0(s)^4|_{s=0} = \exp\Bigl(\frac{1}{3\pi}\bigl(\mathrm{Vol}(\mathcal{M}_\Gamma) - i12\pi^2\eta(D(\sigma_1))\bigr)\Bigr). \tag{3-31}$$

Remark 3.9 For the equality (3-31), the Ruelle zeta function does not appear on the left-hand side. This is because the corresponding term is formally $R_{\rho^{-1}}(0)$, which can be understood to be 1, putting $m=-1$ on the right-hand side of (3-8).

Proof The equality (3-29) for $n=1$ is written as

$$F_1(s)^2 R_{\rho^0}(s)|_{s=0} = \exp\Bigl(-\frac{1}{3\pi}\mathrm{Vol}(\mathcal{M}_\Gamma) - i\pi\eta(D(\sigma_2))\Bigr) \cdot \Bigl(\frac{Z(\sigma_0, -s+2)}{Z(\sigma_0, s+2)} \frac{Z(\sigma_{-2}, s+1)}{Z(\sigma_{-2}, -s+1)}\Bigr)\Bigr|_{s=0}, \tag{3-32}$$

noting $\eta(D(\sigma_0)) = 0$. By (3-28),

$$Z(\sigma_0, \pm s) = \exp\left(\frac{\mathrm{Vol}(\mathcal{M}_\Gamma)}{\pi} \cdot \frac{1}{3}(1 \pm s)^3\right) Z(\sigma_0, 2 \mp s). \tag{3-33}$$

By Theorem 4.6 of [11], the Selberg zeta function $Z(\sigma_0, s)$ has a simple zero at $s = 0$, which corresponds to the zero eigenvalue of the selfadjoint Hodge Laplacian acting on $\Omega^0(\mathcal{M}_\Gamma)$. Combining these, we have

$$\left.\frac{Z(\sigma_0, -s+2)}{Z(\sigma_0, s+2)}\right|_{s=0} = \left.\frac{Z(\sigma_0, s)}{Z(\sigma_0, -s)}\right|_{s=0} = -1. \tag{3-34}$$

Again, by (3-28),

$$Z(\sigma_{-2}, 1+s) = e^{-i\pi\eta(D(\sigma_2))} \exp\left(\frac{\mathrm{Vol}(\mathcal{M}_\Gamma)}{\pi}\left(\tfrac{1}{3}s^3 - s\right)\right) Z(\sigma_2, 1-s), \tag{3-35}$$

noting that $\eta(D(\sigma_2)) = -\eta(D(\sigma_{-2}))$. Hence,

$$\left.\frac{Z(\sigma_{-2}, s+1)}{Z(\sigma_{-2}, -s+1)}\right|_{s=0} = \left.\frac{Z(\sigma_2, -s+1)}{Z(\sigma_{-2}, -s+1)}\right|_{s=0} e^{-i\pi\eta(D(\sigma_2))} = 1. \tag{3-36}$$

The last equality follows from the main theorem of [19, page 2]. By (3-32), (3-34) and (3-36), we conclude that the equality (3-30) holds.

To prove the statement for $G_0(s)$, let us introduce the zeta functions

$$Z^e(s) = Z(\sigma_1, s) Z(\sigma_{-1}, s), \quad Z^o(s) = \frac{Z(\sigma_1, s)}{Z(\sigma_{-1}, s)}. \tag{3-37}$$

By (3-28) and $\eta(D(\sigma_1)) = -\eta(D(\sigma_{-1}))$, we have

$$\begin{aligned} Z^e(1+s) &= \exp\left(\frac{2}{\pi}\mathrm{Vol}(\mathcal{M}_\Gamma)\left(\tfrac{1}{3}s^3 - \tfrac{1}{4}s\right)\right) Z^e(1-s), \\ Z^o(1+s) Z^o(1-s) &= \exp\left(2\pi i \eta(D(\sigma_1))\right). \end{aligned} \tag{3-38}$$

By (3-20), we have

$$\begin{aligned} G_0(s)^2 &= \frac{Z\left(\sigma_{-1}, s+\frac{1}{2}\right)^2}{Z\left(\sigma_1, s+\frac{3}{2}\right)^2} \\ &= \left(\frac{Z\left(\sigma_{-1}, s+\frac{1}{2}\right)}{Z\left(\sigma_1, s+\frac{1}{2}\right)} \cdot \frac{Z\left(\sigma_{-1}, s+\frac{3}{2}\right)}{Z\left(\sigma_1, s+\frac{3}{2}\right)}\right) \cdot \left(\frac{Z\left(\sigma_1, s+\frac{1}{2}\right) Z\left(\sigma_{-1}, s+\frac{1}{2}\right)}{Z\left(\sigma_1, s+\frac{3}{2}\right) Z\left(\sigma_{-1}, s+\frac{3}{2}\right)}\right) \\ &= Z^o\left(s+\tfrac{1}{2}\right)^{-1} Z^o\left(s+\tfrac{3}{2}\right)^{-1} Z^e\left(s+\tfrac{1}{2}\right) Z^e\left(s+\tfrac{3}{2}\right)^{-1} \\ &= Z^o\left(s+\tfrac{1}{2}\right)^{-1} Z^o\left(s+\tfrac{3}{2}\right)^{-1} \frac{Z^e\left(s+\frac{1}{2}\right)}{Z^e\left(-s+\frac{1}{2}\right)} \frac{Z^e\left(-s+\frac{1}{2}\right)}{Z^e\left(s+\frac{3}{2}\right)}. \end{aligned} \tag{3-39}$$

Using this and the equalities in (3-38), we have

$$G_0(s)^2 = \frac{Z^e\bigl(s+\frac{1}{2}\bigr)}{Z^e\bigl(-s+\frac{1}{2}\bigr)} \exp\Bigl(-\frac{2}{\pi}\mathrm{Vol}(\mathcal{M}_\Gamma)\bigl(\tfrac{1}{3}\bigl(s+\tfrac{1}{2}\bigr)^3 - \tfrac{1}{4}\bigl(s+\tfrac{1}{2}\bigr)\bigr)\Bigr) \cdot \exp\bigl(-2\pi i \eta(D(\sigma_1))\bigr). \tag{3-40}$$

Hence, the function $G_0(s)$ is regular at $s=0$. As before, to remove the ambiguity from the first part at $s=0$ on the right-hand side of (3-40), we take its square and obtain the expression of $G_0(s)^4$ at $s=0$ given in (3-31). □

4 Determinant and torsion

4.1 Determinant and Selberg zeta function

We start with:

Lemma 4.1 *For the operator $\Delta_p^\flat$ acting on $\Omega^p(\mathcal{M}_\Gamma, E_\rho)$ with $p=0,1$, we have the asymptotics, as $t \to 0$,*

$$\mathrm{Tr}(e^{-t\Delta_p^\flat}) \sim \frac{1}{4\pi^2}\dim(V_\rho)\,\mathrm{Vol}(\mathcal{M}_\Gamma)(a_1(p)t^{-3/2} + a_2(p)t^{-1/2}) + O(t^{1/2}), \tag{4-1}$$

where $a_1(0) = \frac{1}{2}\sqrt{\pi}$, $a_2(0) = -\frac{1}{2}\sqrt{\pi}$, $a_1(1) = \frac{3}{2}\sqrt{\pi}$ and $a_2(1) = \frac{3}{2}\sqrt{\pi}$.

Proof The small-time asymptotics of $\mathrm{Tr}(e^{-t(\Delta_0^\flat - 1)})$ and $\mathrm{Tr}(e^{-\Delta_1^\flat}) - \mathrm{Tr}(e^{-t\Delta_0^\flat})$ follow from those of the right-hand sides of (2-56) and (2-57), respectively. The second part of the right-hand side, which consists of the sum over the hyperbolic elements, has size $O(e^{-c/t})$ for a constant $c>0$, hence the main contribution is given by the first part from the identity element in Γ. Hence, we have the asymptotics

$$\mathrm{Tr}(e^{-t(\Delta_0^\flat - 1)}) \sim \frac{1}{4\pi^2}\dim(V_\rho)\,\mathrm{Vol}(\mathcal{M}_\Gamma)(a_1(0)t^{-3/2} + a_2(0)t^{-1/2}) + O(t^{1/2}), \tag{4-2}$$

$$\mathrm{Tr}(e^{-\Delta_1^\flat}) - \mathrm{Tr}(e^{-t\Delta_0^\flat}) \sim \frac{1}{4\pi^2}\dim(V_\rho)\,\mathrm{Vol}(\mathcal{M}_\Gamma)(a_1(1)t^{-3/2} + a_2(1)t^{-1/2}) + O(t^{1/2}),$$

where $a_1(0) = \frac{1}{2}\sqrt{\pi}$, $a_2(0) = 0$, $a_1(1) = \sqrt{\pi}$, $a_2(1) = 2\sqrt{\pi}$. From these, the equality (4-1) follows easily. □

Now we choose a complex number s in $\Lambda_\epsilon \setminus B_r$ for some $\epsilon > 0$ and $r>0$ such that the spectrum of $\Delta_p^\flat + s^2$ for $p=0,1$ lies on the right half-plane with its real part bigger than some $\delta > 0$. Under this assumption, for a fixed s, we consider

$$\zeta_p(z,s) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} e^{-ts^2} \operatorname{Tr}(e^{-t\Delta_p^\flat})\, dt. \tag{4-3}$$

Note that this integral converges absolutely and uniformly on compact subsets of the half-plane $\mathrm{Re}(z) > \frac{3}{2}$. Using Lemma 4.1, one can show that $\zeta_p(z,s)$ has a meromorphic extension to $\mathbb{C}$ with respect to z, and it is regular at $z = 0$. For $p = 0, 1$, we define

$$\det(\Delta_p^\flat + s^2) = \exp\Big(-\frac{d}{dz}\Big|_{z=0} \zeta_p(z,s)\Big). \tag{4-4}$$

In a similar way, we define $\det(\Delta_0^\flat - 1 + s^2)$.

Lemma 4.2 *As $s \to \infty$ in the region $\Lambda_\epsilon \setminus B_r$,*

$$\begin{aligned} \log\det(\Delta_0^\flat - 1 + s^2) &= \frac{1}{2\pi} \dim(V_\rho) \operatorname{Vol}(\mathcal{M}_\Gamma)\big(-\tfrac{1}{3}s^3\big) + O(s^{-1}), \\ \log\det(\Delta_1^\flat + s^2) - \log\det(\Delta_0^\flat + s^2) &= \frac{1}{\pi} \dim(V_\rho) \operatorname{Vol}(\mathcal{M}_\Gamma)\big(s - \tfrac{1}{3}s^3\big) + O(s^{-1}). \end{aligned} \tag{4-5}$$

Here we take the principal branch for the logarithm.

Proof The asymptotics as $s \to \infty$ follows from (4-2) and $\int_0^\infty t^{z-1} e^{-ts^2}\, dt = s^{-2z}\Gamma(z)$. □

By Lemma 4.1, we have

$$\begin{aligned} \frac{1}{2s}\frac{d}{ds} \log\det(\Delta_p^\flat + s^2) &- \frac{1}{2s_0}\frac{d}{ds}\Big|_{s=s_0} \log\det(\Delta_p^\flat + s^2) \\ &= \lim_{z\to 0}\Big(-\frac{1}{2s}\frac{d}{ds}(\Gamma(z)\zeta_p(z,s)) + \frac{1}{2s_0}\frac{d}{ds}\Big|_{s=s_0}(\Gamma(z)\zeta_p(z,s))\Big) \\ &= \int_0^\infty (e^{-ts^2} - e^{-ts_0^2}) \operatorname{Tr}(e^{-t\Delta_p^\flat})\, dt. \end{aligned} \tag{4-6}$$

Now, we deal with the geometric side of (2-57) as we did for the spectral side, that is, we multiply e^{-ts^2} to the geometric side of (2-57) and take the integral $\int_0^\infty \cdot\, dt$. First, to deal with the terms from hyperbolic elements, from (2-10) we recall

$$\begin{aligned} D(\gamma)^{-1} &= e^{-\ell_\gamma}(1 - e^{-(\ell_\gamma + i\theta_\gamma)})^{-1}(1 - e^{-(\ell_\gamma - i\theta_\gamma)})^{-1} \\ &= e^{-\ell_\gamma} \sum_{\substack{p\ge 0 \\ q \ge 0}} e^{-p(l_\gamma + i\theta_\gamma)} e^{-q(l_\gamma - i\theta_\gamma)}. \end{aligned} \tag{4-7}$$

Then we obtain the following equalities:

$$\sum_{[\gamma]\neq[e]} \frac{\ell_{\gamma_0} \operatorname{tr}\rho(\gamma)}{D(\gamma)} e^{i\theta_\gamma} \int_0^\infty \frac{1}{\sqrt{4\pi t}} e^{-\ell_\gamma^2/4t - ts^2}\, dt \tag{4-8}$$

$$= \frac{1}{2s} \sum_{[\gamma]\neq[e]} \frac{\ell_{\gamma_0} \operatorname{tr}\rho(\gamma)}{D(\gamma)} e^{i\theta_\gamma} e^{-s\ell_\gamma}$$

$$= \frac{1}{2s} \sum_{[\gamma]\neq[e]} \ell_{\gamma_0} \operatorname{tr}\rho(\gamma) e^{i\theta_\gamma} \sum_{\substack{p\geq 0\\ q\geq 0}} e^{-p(l_\gamma+i\theta_\gamma)} e^{-q(l_\gamma-i\theta_\gamma)} e^{-(s+1)\ell_\gamma}$$

$$= \frac{1}{2s} \sum_{[\gamma]_{\mathrm{p}}} \sum_{\substack{p\geq 0\\ q\geq 0}} \sum_{m=1}^{\infty} \frac{\ell_\gamma}{m} (\operatorname{tr}\rho(\gamma) e^{i\theta_\gamma} e^{-p(l_\gamma+i\theta_\gamma)} e^{-q(l_\gamma-i\theta_\gamma)} e^{-(s+1)\ell_\gamma})^m$$

$$= -\frac{1}{2s} \sum_{[\gamma]_{\mathrm{p}}} \sum_{\substack{p\geq 0\\ q\geq 0}} \ell_\gamma \log\det(\mathrm{Id}_{V_\rho} - \rho(\gamma) e^{i\theta_\gamma} e^{-p(l_\gamma+i\theta_\gamma)} e^{-q(l_\gamma-i\theta_\gamma)} e^{-(s+1)\ell_\gamma})$$

$$= \frac{1}{2s} \frac{d}{ds} \log Z_\rho(\sigma_2, s+1).$$

Here the sums in the third and fourth lines run over the set of conjugacy classes in Γ of *primitive* hyperbolic elements in Γ. For the first equality above, we used

$$\int_0^\infty \frac{1}{\sqrt{4\pi t}} e^{-\ell^2/4t - ts^2}\, dt = \frac{1}{2s} e^{-s\ell}. \tag{4-9}$$

Repeating the same procedure for the identity contribution of the geometric side of (2-57),

$$\frac{1}{2\pi^2} \dim(V_\rho) \operatorname{Vol}(\mathcal{M}_\Gamma) \int_0^\infty e^{-ts^2} \int_0^\infty (\lambda^2+1) e^{-t\lambda^2}\, d\lambda\, dt \tag{4-10}$$

$$= \frac{1}{2\pi^2} \dim(V_\rho) \operatorname{Vol}(\mathcal{M}_\Gamma) \int_0^\infty e^{-ts^2} \sqrt{\pi} \bigl(\tfrac{1}{2} t^{-3/2} + t^{-1/2}\bigr)\, dt$$

$$= \frac{1}{2s} \frac{1}{\pi} \dim(V_\rho) \operatorname{Vol}(\mathcal{M}_\Gamma)(-s^2+1).$$

Combining (2-57), (4-6), (4-8) and (4-10),

$$\frac{1}{s}\frac{d}{ds} \log \frac{\det(\Delta_1^\flat + s^2)}{\det(\Delta_0^\flat + s^2)} - \frac{1}{s_0} \frac{d}{ds}\Big|_{s=s_0} \log \frac{\det(\Delta_1^\flat + s^2)}{\det(\Delta_0^\flat + s^2)} \tag{4-11}$$

$$= \frac{1}{s}\frac{d}{ds} \log(Z_\rho(\sigma_2, s+1) Z_\rho(\sigma_{-2}, s+1))$$

$$- \frac{1}{s_0}\frac{d}{ds}\Big|_{s=s_0} \log(Z_\rho(\sigma_2, s+1) Z_\rho(\sigma_{-2}, s+1))$$

$$+ \frac{1}{s}\frac{1}{\pi} \dim(V_\rho) \operatorname{Vol}(\mathcal{M}_\Gamma)(-s^2+1) - \frac{1}{s_0}\frac{1}{\pi} \dim(V_\rho) \operatorname{Vol}(\mathcal{M}_\Gamma)(-s_0^2+1),$$

which implies

$$\begin{aligned} \text{(4-12)} \quad & \log\det(\Delta_1^\flat + s^2) - \log\det(\Delta_0^\flat + s^2) \\ &= \log Z_\rho(\sigma_2, s+1) + \log Z_\rho(\sigma_{-2}, s+1) \\ &\qquad + \frac{1}{\pi}\dim(V_\rho)\operatorname{Vol}(\mathcal{M}_\Gamma)\bigl(s - \tfrac{1}{3}s^3\bigr) + c_2 s^2 + c_0 \end{aligned}$$

for some constants c_2 and c_0. In a similar way, using (2-56), we obtain

$$\begin{aligned} \text{(4-13)} \quad & \log\det(\Delta_0^\flat - 1 + s^2) \\ &= \log Z_\rho(\sigma_0, s+1) + \frac{1}{2\pi}\dim(V_\rho)\operatorname{Vol}(\mathcal{M}_\Gamma)\bigl(-\tfrac{1}{3}s^3\bigr) + d_2 s^2 + d_0 \end{aligned}$$

for some constants d_2 and d_0. By Lemma 4.2 and the fact that $\log Z_\rho(\sigma_k, s)$ decays exponentially as $s \to \infty$, the constants c_2, c_0, d_2 and d_0 are trivial. Hence, we have:

Proposition 4.3 *For $s \in \mathbb{C}$,*

$$\text{(4-14)} \qquad \det(\Delta_0^\flat - 1 + s^2) = Z_\rho(\sigma_0, s+1)\exp\Bigl(-\frac{1}{6\pi}\dim(V_\rho)\operatorname{Vol}(\mathcal{M}_\Gamma)s^3\Bigr),$$

and

$$\begin{aligned} \text{(4-15)} \quad \frac{\det(\Delta_1^\flat + s^2)}{\det(\Delta_0^\flat + s^2)} &= Z_\rho(\sigma_2, s+1) Z_\rho(\sigma_{-2}, s+1) \\ &\qquad \cdot \exp\Bigl(\frac{1}{\pi}\dim(V_\rho)\operatorname{Vol}(\mathcal{M}_\Gamma)\bigl(s - \tfrac{1}{3}s^3\bigr)\Bigr). \end{aligned}$$

Proof Although the left-hand sides of (4-14) and (4-15) are defined a priori for s with some conditions, these can be extended over $\mathbb{C}$ by the meromorphicity of $Z_\rho(\sigma_k, s)$. □

By (3-14), (4-14) and (4-15),

$$\begin{aligned} \text{(4-16)} \quad R_\rho(s) &= \frac{Z_\rho(\sigma_0, s) Z_\rho(\sigma_0, s+2)}{Z_\rho(\sigma_2, s+1) Z_\rho(\sigma_{-2}, s+1)} \\ &= \frac{\det(\Delta_0^\flat - 1 + (s-1)^2)\det(\Delta_0^\flat - 1 + (s+1)^2)\det(\Delta_0^\flat + s^2)}{\det(\Delta_1^\flat + s^2)} \\ &\qquad \cdot \exp\Bigl(\frac{2s}{\pi}\dim(V_\rho)\operatorname{Vol}(\mathcal{M}_\Gamma)\Bigr). \end{aligned}$$

This equality implies the following functional equation of $R_\rho(s)$:

Theorem 4.4 *For $s \in \mathbb{C}$,*

$$\text{(4-17)} \qquad R_\rho(s) = R_\rho(-s)\exp\Bigl(\frac{4s}{\pi}\dim(V_\rho)\operatorname{Vol}(\mathcal{M}_\Gamma)\Bigr).$$

Recalling (2-60), equation (4-17) is compatible with Theorem 1.1 of [11], which holds for unitary representations ρ. Note that the Ruelle zeta function in this paper is the inverse of the one of [11]. A similar formula to (4-17) was also given in (3.11) of [23].

4.2 Reidemeister torsion

For a k–dimensional vector space over $\mathbb{C}$, let $v=(v_1,\ldots,v_k)$ and $w=(w_1,\ldots,w_k)$ be two bases for V. Let $[w/v]$ denote the determinant of the matrix $T=(t_{ij})$ representing the change of base from v to w, that is, $w_i=\sum t_{ij}v_j$. Suppose

$$C\colon C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \tag{4-18}$$

is a chain complex of finite complex modules. Let Z_q denote the kernel of ∂ in C_q, $B_q\subset C_q$ the image of C_{q+1} under ∂, and $H_q(C)=Z_q/B_q$ the q^{th} homology group of C. Choose a base b_q for B_q for each q, and let $\widetilde{b}_{q-1}$ be an independent set in C_q such that $\partial\widetilde{b}_{q-1}=b_{q-1}$, and $\widetilde{h}_q$ an independent set in Z_q representing a base h_q of $H_q(C)$. Then $(b_q,\widetilde{h}_q,\widetilde{b}_{q-1})$ is a base for C_q. For a given preferred base c_q for C_q, note that $[b_q,\widetilde{h}_q,\widetilde{b}_{q-1}/c_q]$ depends only on b_q, h_q and b_{q-1}, hence we denote it by $[b_q,h_q,b_{q-1}/c_q]$. The *torsion* $\tau(C)$ of the chain complex C is the nonzero complex number defined by

$$\mathcal{T}(C)=\prod_{q=0}^{n}[b_q,h_q,b_{q-1}/c_q]^{(-1)^q}. \tag{4-19}$$

Note that $\mathcal{T}(C)$ depends only on the choice of the bases c_q and h_q, and not on the choice of the base b_q.

Let K be a finite cell complex and $\widetilde{K}$ the simply connected covering space of K with the fundamental group π_1 of K acting as deck transformations on $\widetilde{K}$. Since $\widetilde{K}$ is just the set of translates of a fundamental domain under π_1, the complex cochain groups $C^q(\widetilde{K})$ become modules over the complex group algebra $\mathbb{C}(\pi_1)$ with a preferred base consisting of the dual element of cells of K. Relative to this preferred base, the boundary operator on the right $\mathbb{C}(\pi_1)$–module $C^q(\widetilde{K})$ is a matrix with coefficients in $\mathbb{C}(\pi_1)$. For a representation ρ of $\pi_1(K)$ into $\mathrm{SL}(N,\mathbb{C})$, define the chain complex $C(K,\rho)$ by

$$C^q(K,\rho)=C^q(\widetilde{K})\otimes_{\mathbb{C}(\pi_1)}\mathbb{C}^N, \tag{4-20}$$

where $\mathbb{C}^N$ is considered as a left $\mathbb{C}(\pi_1)$–module via the action of ρ. The boundary map of $C(K,\rho)$ is defined to be the dual map of the boundary map of the cell complex.

We choose a preferred base $e_i \otimes x_j$, where e_i runs through the preferred base of $C^q(\widetilde{K})$ and x_j runs through a base for $\mathbb{C}^N$. Now the *Reidemeister torsion* $\mathcal{T}(K,\rho)$ attached to the representation ρ is defined by

$$\mathcal{T}(K,\rho) = \mathcal{T}(C(K,\rho)). \tag{4-21}$$

A different choice of the preferred base e_i can give at most a sign change of $\mathcal{T}(K,\rho)$ since ρ is a representation into $\mathrm{SL}(N,\mathbb{C})$. A different choice of the base x'_j for $\mathbb{C}^N$ can also give a change by the factor $[x'/x]^{\chi(C)}$, where $\chi(C)$ denotes the Euler characteristic of C. Hence, if $\chi(C) = 0$, the Reidemeister torsion $\mathcal{T}(C(K,\rho))$ is well defined as an invariant with a value in $\mathbb{C}^*/\{\pm 1\}$ depending only on the choice of the base h_q for $H_q(C)$. By [20], it is known that $\mathcal{T}(C(K,\rho))$ is a combinatorial invariant of (K,ρ). Hence, if $\mathcal{M}$ is a compact oriented manifold, any smooth triangulation of $\mathcal{M}$ gives the same Reidemeister torsion. We denote it by $\mathcal{T}(\mathcal{M},\rho)$.

When the cohomology groups $H^*(\mathcal{M},\rho) := H_*(C(K,\rho))$ are trivial for a smooth triangulation K of $\mathcal{M}$, the square of Reidemeister torsion $\mathcal{T}^2(\mathcal{M},\rho)$ is a well-defined complex number.

To state Theorem 10.1 of [7], now we introduce the analytic torsion defined by the nonselfadjoint operators $\Delta_p^\flat$ in the context of Section 2.4. Recall that the operator $\Delta_p^\flat$ acts on $C^\infty(\mathcal{M}, E_\rho)$, where $\mathcal{M}$ is a closed Riemannian manifold of dimension n and E is a flat vector bundle defined by a representation $\rho\colon \pi_1(\mathcal{M}) \to \mathrm{SL}(N,\mathbb{C})$. In general, the nonselfadjoint operator $\Delta_p^\flat$ acting on $\Omega^p(\mathcal{M}, E_\rho)$ may have a generalized eigenvalue with nonpositive real part. Hence, the definitions (4-3) and (4-4) do not work simply if we put $s = 0$ at (4-3) and (4-4). For this, we recall the construction in [7, Section 8]. Let $r > 0$ be a real number that is not the real part of any generalized eigenvalue of $\Delta_p^\flat$. Let $\Pi_{p,r}$ denote the spectral projection on the span of the generalized eigenvectors with generalized eigenvalues with real part less than r. Noting that Lemma 4.1 still holds for the heat operator of $\Delta_{p,r}^\flat := (\mathrm{Id} - \Pi_{p,r})\Delta_p^\flat$ since $\Pi_{p,r}$ is a smoothing operator, we define

$$\det \Delta_{p,r}^\flat = \exp\Big(-\frac{d}{dz}\Big|_{z=0} \zeta_{p,r}(z)\Big), \tag{4-22}$$

where

$$\zeta_{p,r}(z) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1}\,\mathrm{Tr}(e^{-t\Delta_{p,r}^\flat})\,dt. \tag{4-23}$$

Let us denote the zero generalized eigenspace of $\Delta_p^\flat$ by $\Omega_0^p(\mathcal{M}, E_\rho)$. The cohomology of the complex $(\Omega_0^*(\mathcal{M}, E_\rho), d)$ is the same as the cohomology $H^*(\mathcal{M}, E_\rho)$ of

$(\Omega^*(\mathcal{M}, E_\rho), d)$. We define its torsion as in (4-19) by

$$\mathcal{T}_0(\mathcal{M}, \rho) = \mathcal{T}(\Omega_0^*(\mathcal{M}, E_\rho), d). \tag{4-24}$$

When $H^*(\mathcal{M}, E_\rho)$ is nontrivial, we need to specify a basis of $H^*(\mathcal{M}, E_\rho)$ in the above definition (4-24). But, since the case we want to consider satisfies the acyclic condition, we assume the acyclic condition from now on. Let us remark that the multiplicity of the zero generalized eigenvalue of $\Delta_p^\flat$ can be nontrivial under the acyclic condition in general. Now the *analytic torsion* $T_a(\mathcal{M}, \rho)$ is defined by

$$\begin{aligned} T_a(\mathcal{M}, \rho) \\ = \mathcal{T}_0(\mathcal{M}, \rho)^2 \cdot \prod_{p=1}^{n} (\det \Delta_{p,r}^\flat)^{p(-1)^{p+1}} \cdot \prod_{p=1}^{n} \Bigl(\prod_{\lambda_{p,j} \in S(p,r)} \lambda_{p,j} \Bigr)^{p(-1)^{p+1}}. \end{aligned} \tag{4-25}$$

Here $S(p, r)$ denotes the set of all the nonzero generalized eigenvalues with real part less than r counted with multiplicities. By Theorem 8.3 of [7], the above definition of $T_a(\mathcal{M}, \rho)$ in (4-25) does not depend on the choice of r. When the cohomology groups $H^*(\mathcal{M}, E_\rho)$ are trivial, the following equality holds between two complex-valued invariants:

$$\mathcal{T}^2(\mathcal{M}, \rho) = T_a(\mathcal{M}, \rho) \tag{4-26}$$

by Theorem 10.1 of [7].

5 The case of compact hyperbolic 3–manifolds

By Remark 3.7, $R_{\rho^m}(s)$ is regular and has a finite nonzero value at $s = 0$ for $m \geq 3$. By this fact and (4-16), we have

$$R_{\rho^m}(0) = \lim_{s \to 0} \frac{\det(\Delta_0^\flat + s^2 - 2s) \det(\Delta_0^\flat + s^2 + 2s) \det(\Delta_0^\flat + s^2)}{\det(\Delta_1^\flat + s^2)}. \tag{5-1}$$

For $p = 0, 1$, we take a sufficiently small $r > 0$ such that the real parts of the generalized eigenvalues of $\Delta_p^\flat$ with positive real parts are bigger than r. Then, it is easy to check that

$$\begin{aligned} \lim_{s \to 0} s^{-2h_p} \det(\Delta_p^\flat + s^2) &= \lim_{s \to 0} \det(\Delta_{p,r}^\flat + s^2) \cdot s^{-2h_p} \prod_{\lambda_{p,j} \in S(p,r)} (\lambda_{p,j} + s^2) \\ &= \det \Delta_{p,r}^\flat \cdot \prod_{\lambda_{p,j} \in S(p,r)} \lambda_{p,j}, \end{aligned} \tag{5-2}$$

where h_p denotes the multiplicities of the zero generalized eigenvalue of $\Delta_p^\flat$. Similar equalities hold for the other factors $\det(\Delta_0^\flat + s^2 \pm 2s)$ on the right-hand side of (5-1). By (5-1), we have $2h_0 = h_1$. Using this and by (4-25), (5-1) and (5-2),

$$\begin{aligned}
\text{(5-3)} \quad R_{\rho^m}(0) &= \lim_{s\to 0} \frac{s^{-4h_0}\det(\Delta_0^\flat+s^2-2s)\det(\Delta_0^\flat+s^2+2s)\det(\Delta_0^\flat+s^2)}{s^{-2h_1}\det(\Delta_1^\flat+s^2)} \\
&= (-4)^{h_0}\frac{\bigl(\det\Delta_{0,r}^\flat\cdot\prod_{\lambda_{0,j}\in S(0,r)}\lambda_{0,j}\bigr)^3}{\det\Delta_{1,r}^\flat\cdot\prod_{\lambda_{1,j}\in S(1,r)}\lambda_{1,j}} \\
&= (-4)^{h_0}\frac{\bigl(\det\Delta_{1,r}^\flat\cdot\prod_{\lambda_{1,j}\in S(1,r)}\lambda_{1,j}\bigr)\bigl(\det\Delta_{3,r}^\flat\cdot\prod_{\lambda_{3,j}\in S(3,r)}\lambda_{3,j}\bigr)^3}{\bigl(\det\Delta_{2,r}^\flat\cdot\prod_{\lambda_{2,j}\in S(2,r)}\lambda_{2,j}\bigr)^2} \\
&= (-4)^{h_0}\frac{T_a(\mathcal{M}_\Gamma,\rho^m)}{\mathcal{T}_0(\mathcal{M}_\Gamma,\rho^m)^2}.
\end{aligned}$$

For the third equality of (5-3), we used

$$\text{(5-4)} \qquad (\star\otimes \mathrm{Id}_{E_{\rho^m}})\Delta_p^\flat = \Delta_{3-p}^\flat(\star\otimes\mathrm{Id}_{E_{\rho^m}}) \quad \text{over } \Omega^p(\mathcal{M}_\Gamma, E_{\rho^m}),$$

which follows from (2-27).

For the representation $\rho^m\colon \Gamma \to \mathrm{SL}(S^m(\mathbb{C}^2)) \subset \mathrm{SL}(\mathbb{C}^{m+1})$, the cohomology groups $H^*(\mathcal{M}_\Gamma, E_{\rho^m})$ are trivial by Theorem 6.7 in [5]. Hence, the square of the Reidemeister torsion $\mathcal{T}^2(\mathcal{M}_\Gamma,\rho^m)$ is a well-defined complex-valued invariant. By (4-26), we have $\mathcal{T}^2(\mathcal{M}_\Gamma,\rho^m) = T_a(\mathcal{M}_\Gamma,\rho^m)$. Combining this and (5-3),

$$\text{(5-5)} \qquad (-4)^{h_0}\frac{\mathcal{T}(\mathcal{M}_\Gamma,\rho^m)^2}{\mathcal{T}_0(\mathcal{M}_\Gamma,\rho^m)^2} = R_{\rho^m}(0).$$

Actually, we can improve the equality (5-5) by the following proposition:

Proposition 5.1 *For the terms h_0 and $\mathcal{T}_0(\mathcal{M}_\Gamma,\rho^m)$ appearing in* (5-5), *we have*

$$\text{(5-6)} \qquad \begin{aligned} h_0 = 1 &\quad \textit{and} \quad |\mathcal{T}_0(\mathcal{M}_\Gamma,\rho^m)| = 2 \quad \textit{if } m \textit{ is even}, \\ h_0 = 0 &\quad \textit{and} \quad \mathcal{T}_0(\mathcal{M}_\Gamma,\rho^m) = 1 \quad \textit{if } m \textit{ is odd}. \end{aligned}$$

Proof First, we recall that there exists a canonical Hermitian metric over each fiber of E_{ρ^m} constructed in Lemma 3.1 in [15], which is called an *admissible metric*. Using the hyperbolic metric over $\mathcal{M}_\Gamma$ and this admissible metric over E_{ρ^m}, one can define the Laplacians Δ_p acting on $\Omega^p(\mathcal{M}_\Gamma, E_{\rho^m})$ for $p = 0, 1, 2, 3$, which are selfadjoint nonnegative operators. Then the space of zero eigensections of Δ_p is the same as

$H^p(\mathcal{M}_\Gamma, E_{\rho^m})$ by Theorem 6.1 in [15], which vanishes by Theorem 6.7 in [5]. By Theorem 1.5 of [23], we have

$$|R_{\rho^m}(0)| = \mathrm{T}_a(\mathcal{M}_\Gamma, \rho^m), \tag{5-7}$$

where $\mathrm{T}_a(\mathcal{M}_\Gamma, \rho^m)$ denotes the analytic torsion defined by the selfadjoint Laplacians Δ_p for $p = 0, 1, 2, 3$. This was also proved in [36]. The proof in [23] is along the same lines as in the previous sections for the selfadjoint Laplacian Δ_p. By [4; 21], we also have

$$\mathrm{T}_a(\mathcal{M}_\Gamma, \rho^m) = |\mathcal{T}(\mathcal{M}_\Gamma, \rho^m)|^2. \tag{5-8}$$

Combining (5-7) and (5-8), we have

$$|R_{\rho^m}(0)| = |\mathcal{T}(\mathcal{M}_\Gamma, \rho^m)|^2. \tag{5-9}$$

Secondly, the multiplicity h_0 is the same as the degree of the zero of $Z_{\rho^m}(\sigma_0, s)$ at $s=0$ by (4-14). This is also the same as the degree of the zero of $\prod_{l=0}^m Z\big(\sigma_{m-2l}, s-\frac{1}{2}m+l\big)$ at $s = 0$ by (3-7). When m is even, there is the factor $Z(\sigma_0, s)$, which has a simple zero at $s = 0$. Other factors consist of $Z\big(\sigma_k, s - \frac{1}{2}k\big)$ for $k = m-2l$ with $0 \le l \le m$ and $l \ne \frac{1}{2}m$. By the construction of [23, Section 6], there exists a second-order elliptic differential operator $\Delta(\sigma_k)$ acting on $C^\infty(\mathcal{M}_\Gamma, E(\sigma_k))$. Here $E(\sigma_k)$ is a locally homogeneous vector bundle over $\mathcal{M}_\Gamma$ defined by

$$E(\sigma_k) := E_{\tau_{|k|}} \otimes E^*_{\tau_{|k|-2}} \otimes E_{\tau_{|k|-4}} \otimes \cdots \tag{5-10}$$

using (2-16) and (2-18), where $E^*_{\tau_{|k|-2}}$ denotes the dual bundle of $E_{\tau_{|k|-2}}$ and the tensor product ends with (the dual of) E_{τ_1} or E_{τ_0} on the right-hand side of (5-10). Hence the vector bundle $E(\sigma_k)$ is naturally equipped with a $\mathbb{Z}_2$–grading. By the equality below (6.17) in [23],

$$Z\big(\sigma_k, s - \tfrac{1}{2}k\big) Z\big(\sigma_{-k}, s - \tfrac{1}{2}k\big) = \det\nolimits_{\mathrm{gr}}(\Delta(\sigma_k) + s(s-k-2)) \exp(P_k(s)) \tag{5-11}$$

for a polynomial $P_k(s)$. Here $\det_{\mathrm{gr}}$ denotes the ratio of the usual regularized determinants of the restrictions of the operator $\Delta(\sigma_k)$ to the subspaces of $C^\infty(\mathcal{M}_\Gamma, E(\sigma_k))$ defined by the grading of the vector bundle $E(\sigma_k)$. Since this operator has trivial kernel by Lemma 7.2 in [23], one can conclude that $Z\big(\sigma_k, s-\frac{1}{2}k\big) Z\big(\sigma_{-k}, s-\frac{1}{2}k\big)$ has no zero at $s = 0$ by (5-11). Since $\big|Z\big(\sigma_k, s - \frac{1}{2}k\big)\big|^2 = Z\big(\sigma_k, s - \frac{1}{2}k\big) Z\big(\sigma_{-k}, s - \frac{1}{2}k\big)$ for real s, the same is true for $Z\big(\sigma_k, s - \frac{1}{2}k\big)$ for $k = m-2l$ with $0 \le l \le m$ and $l \ne \frac{1}{2}m$. Hence, $h_0 = 1$ if m is even and $h_0 = 0$ if m is odd.

Finally, comparing (5-5) with (5-9), one obtains $|\mathcal{T}_0(\mathcal{M}_\Gamma, \rho^m)| = 2$ when m is even. When m is odd, $h_1 = 2h_0 = 0$. This also imply the multiplicities of zero generalized eigenvalues of $\Delta_p^\flat$ vanish for $p = 2, 3$ by (5-4). Hence, we have $\mathcal{T}_0(\mathcal{M}_\Gamma, \rho^m) = 1$ when m is odd. □

From now on we split the proof into when m is even or odd.

When $m = 2(n-1)$ with $n \geq 3$, by (3-25), (5-5) and Proposition 5.1,

$$\left(\frac{1}{2}\frac{\mathcal{T}_0(\mathcal{M}_\Gamma, \rho^{2(n-1)})}{\mathcal{T}(\mathcal{M}_\Gamma, \rho^{2(n-1)})}\right)^4 = \exp\bigl(2\pi i\bigl(\eta(D(\sigma_{2n})) - \eta(D(\sigma_{2(n-1)}))\bigr)\bigr) \cdot \exp\Bigl(\frac{2}{\pi}\bigl(2n^2 - 2n + \tfrac{1}{3}\bigr)\mathrm{Vol}(\mathcal{M}_\Gamma)\Bigr) \cdot F_n(\mathcal{M}_\Gamma)^4. \tag{5-12}$$

Here the Zograf infinite product $F_n(\mathcal{M}_\Gamma)$ is defined by

$$F_n(\mathcal{M}_\Gamma) := F_n(0) = \prod_{k=n}^{\infty} R(\sigma_{-2k}, k) = \prod_{k=n}^{\infty}\prod_{[\gamma]_{\mathrm{p}}}(1 - e^{-k(\ell_\gamma + i\theta_\gamma)}). \tag{5-13}$$

Recall that $s = 0$ lies in the convergence domain of $F_n(s)$ for $n \geq 3$ (see (3-17)). By (2-23) and (5-12), we have:

Theorem 5.2 *For a closed hyperbolic* 3*–manifold* $\mathcal{M}_\Gamma$ *defined by a cocompact torsion-free discrete subgroup* $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$, *for* $n \geq 3$,

$$\left(\frac{1}{2}\frac{\mathcal{T}_0(\mathcal{M}_\Gamma, \rho^{2(n-1)})}{\mathcal{T}(\mathcal{M}_\Gamma, \rho^{2(n-1)})}\right)^{12} = \exp(6\pi i\theta_{2(n-1)}) \exp\Bigl(\frac{2}{\pi}(6n^2 - 6n + 1)\mathbb{V}(\mathcal{M}_\Gamma)\Bigr) F_n(\mathcal{M}_\Gamma)^{12}, \tag{5-14}$$

where

$$\theta_{2(n-1)} = \eta(D(\sigma_{2n})) - \eta(D(\sigma_{2(n-1)})) - (6n^2 - 6n + 1)\eta(D(\sigma_2)). \tag{5-15}$$

Let us remark that $\theta_0 = 0$ with $n = 1$ by the definition as expected from (3-30). Hence, $\theta_{2(n-1)}$ can be understood as an anomaly for nonzero $m = 2(n-1)$.

When $m = 2n-1$ with $n \geq 2$, by (3-26), (5-5) and Proposition 5.1,

$$\mathcal{T}(\mathcal{M}_\Gamma, \rho^{2n-1})^{-4} = \exp\bigl(2\pi i\bigl(\eta(D(\sigma_{2n+1})) - \eta(D(\sigma_{2n-1}))\bigr)\bigr) \cdot \exp\Bigl(\frac{2}{\pi}\bigl(2n^2 - \tfrac{1}{6}\bigr)\mathrm{Vol}(\mathcal{M}_\Gamma)\Bigr) \cdot G_n(\mathcal{M}_\Gamma)^4. \tag{5-16}$$

Here the Zograf infinite product $G_n(\mathcal{M}_\Gamma)$ is defined by

$$G_n(\mathcal{M}_\Gamma) := G_n(0) = \prod_{k=n}^{\infty} R\big(\sigma_{-(2k+1)}, k+\tfrac{1}{2}\big) = \prod_{k=n}^{\infty} \prod_{[\gamma]_\mathrm{p}} (1 - e^{-(k+1/2)(\ell_\gamma + i\theta_\gamma)}). \tag{5-17}$$

Recall that $s=0$ lies in the convergence domain of $G_n(s)$ for $n \geq 2$. Let us remark that the equality (5-16) is compatible with (3-31) if we put $n=0$ formally and note $\eta(D(\sigma_1)) = -\eta(D(\sigma_{-1}))$ (see Remark 3.9). Rewriting (5-16) as above, we obtain:

Theorem 5.3 *For a closed hyperbolic* 3*–manifold* $\mathcal{M}_\Gamma$ *defined by a cocompact torsion-free discrete subgroup* $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$*, for* $n \geq 2$*,*

$$\mathcal{T}(\mathcal{M}_\Gamma, \rho^{2n-1})^{-12} = \exp(6\pi i\,\theta_{2n-1}) \cdot \exp\Big(\frac{2}{\pi}\big(6n^2 - \tfrac{1}{2}\big)\mathbb{V}(\mathcal{M}_\Gamma)\Big) \cdot G_n(\mathcal{M}_\Gamma)^{12}, \tag{5-18}$$

where $\theta_{2n-1} := \eta(D(\sigma_{2n+1})) - \eta(D(\sigma_{2n-1})) - \big(6n^2 - \tfrac{1}{2}\big)\eta(D(\sigma_2))$*.*

Part II Noncompact case

The purpose of Part II is to prove Theorems 1.1 and 8.1 for hyperbolic 3–manifolds with cusps. For this, we apply the prior results to a sequence of compact hyperbolic 3–manifolds obtained by Dehn surgeries from a given hyperbolic 3–manifold with cusps.

6 Basic materials, II

6.1 Deformation space of hyperbolic structures

Suppose that $\mathcal{M}_0$ is a complete hyperbolic 3–manifold of finite volume with h cusps. Then $\mathcal{M}_0$ has an ideal triangulation

$$\mathcal{M}_0 = \Delta(z_1^0) \cup \cdots \cup \Delta(z_n^0).$$

Here $\Delta(z_i^0)$ is an ideal tetrahedron described (up to isometry) by the complex number z_i^0 in the upper half-plane such that the Euclidean triangle cut out of any vertex of $\Delta(z_i^0)$ by a horosphere section is similar to the triangle with vertexes 0, 1 and z_i^0. If we deform $(z_1^0, \dots, z_n^0)$ to $(z_1, \dots, z_n)$ slightly with $\operatorname{Im} z_i > 0$ for $i = 1, \dots, n$, then we obtain a complex $\Delta(z_1)\cup\cdots\cup\Delta(z_n)$ with the same gluing pattern as $\mathcal{M}_0$. The necessary and sufficient condition for $\Delta(z_1)\cup\cdots\cup\Delta(z_n)$ to give a smooth (not necessarily complete) hyperbolic manifold is that at each edge e of $\Delta(z_1)\cup\cdots\cup\Delta(z_n)$ the tetrahedron $\Delta(z_i)$ abutting e close up as one goes around e, and thus the product of the corresponding moduli of $\Delta(z_i)$ at e be $\exp(2\pi i)$ (the product is taken in the universal cover of $\mathbb{C}^*$). The consistency condition at e is written as

$$\prod_{i=1}^{n} z_i^{r_i}(1-z_i)^{r_i'} = \pm 1$$

for some integers r_i and r_i' depending on e. Once we have chosen the numbers z_i satisfying the consistency conditions, $\Delta(z_1)\cup\cdots\cup\Delta(z_n)$ acquires a smooth hyperbolic structure, in general incomplete. The deformation space $\mathscr{D}(\mathcal{M}_0)$ of the hyperbolic structures on the underlying topological manifold of $\mathcal{M}_0$ is the variety of $z=(z_1,\dots,z_n)\in\mathbb{C}^n$ which satisfies the consistency conditions.

Choose a pair of simple closed curves (m_i,l_i) on each torus section T_i of the i^{th} cusp which forms a basis of $H_1(T_i,\mathbb{Z})$. For each $z=(z_1,\dots,z_n)\in\mathscr{D}(\mathcal{M}_0)$, let $\rho_z\colon \pi_1(\mathcal{M}_0)\to \mathrm{PSL}(2,\mathbb{C})$ be a holonomy representation of the corresponding (in)complete hyperbolic manifold $\Delta(z_1)\cup\cdots\cup\Delta(z_n)$. We may consider (m_i,l_i) as elements of $\pi_1(\mathcal{M}_0)$. If $\rho_z(m_i)$ and $\rho_z(l_i)$ are not parabolic, they have two fixed points in $\mathbb{C}\cup\{\infty\}$, which we can put at 0 and ∞, so, as Möbius transformations on $\mathbb{C}\cup\{\infty\}$,

$$\rho_z(m_i)\colon w\to a_i w,\quad \rho_z(l_i)\colon w\to b_i w$$

for some $a_i,b_i\in\mathbb{C}^*$. Set $u_i=\log a_i$ and $v_i=\log b_i$. If $\rho_z(m_i)$ and $\rho_z(l_i)$ are parabolic, we set $u_i=v_i=0$. By [34; 26], we have:

Theorem 6.1 (Thurston [34]; Neumann and Zagier [26]) *The deformation space $\mathscr{D}(\mathcal{M}_0)$ of hyperbolic structures on the underlying topological manifold of $\mathcal{M}_0$ can be holomorphically parametrized by $(u_1,\dots,u_h)\in\mathbb{C}^h$ in a neighborhood V of the origin $0=(0,\dots,0)$ in $\mathscr{D}(\mathcal{M}_0)$. For $i=1,\dots,h$, there are holomorphic functions $\tau_i(u)$ over V such that $v_i=\tau_i(u)u_i$ and $\tau_i(0)$ is in the upper half-plane and is the modulus of the Euclidean structure on the torus section T_i associated to the i^{th} cusp of $\mathcal{M}_0$ (with respect to m_i,l_i).*

We denote by $\mathcal{M}_u$ the (in)complete hyperbolic 3–manifold corresponding to the point $u=(u_1,\dots,u_h)\in\mathscr{D}(\mathcal{M}_0)$.

By the second statement in Theorem 6.1, if u is near the origin and $u_i \neq 0$, then v_i is not a real multiple of u_i. Hence there is a unique solution $(p_i, q_i) \in \mathbb{R}^2 \cup \{\infty\}$ to

$$p_i u_i + q_i v_i = 2\pi i. \tag{6-1}$$

We take $(p_i, q_i) = \infty$ if $u_i = 0$. This (p_i, q_i) is called the generalized *Dehn surgery coefficient* by Thurston [34]. If each (p_i, q_i) is a pair of coprime integers, $\mathcal{M}_u$ can be completed to a closed hyperbolic manifold, denoted by $\mathcal{M}_{p,q}$, where $p = (p_1, \dots, p_h)$ and $q = (q_1, \dots, q_h)$, by (p_i, q_i)–hyperbolic Dehn surgery to each end of $\mathcal{M}_u$.

6.2 Invariants over $\mathscr{D}(\mathcal{M}_0)$

For $u \in \mathscr{D}(\mathcal{M}_0)$, the construction of the complex 3–form C in Section 2.3 can be repeated to have a complex 3–form C on the frame bundle $F(\mathcal{M}_u)$. We refer to [37, Section 3] for more details about this construction. Now let s_u be the section defined by an orthonormal framing $\mathcal{F}_u$ on a subset of $\mathcal{M}_u$ such that $s_u^* C$ vanishes over h ends of $\mathcal{M}_u$. It is called the *simple framing* by Yoshida [37]. Since s_u satisfies this vanishing condition over the ends, there is an obstruction for s_u to be defined over whole $\mathcal{M}_u$, which is given by a link L inside of $\mathcal{M}_u$. Hence, s_u is a section from $\mathcal{M}_u \setminus L$ to $F(\mathcal{M}_u)$. Let κ_u be an orthonormal framing over a tubular neighborhood of L such that its first component is tangent to L and has the same direction as the first component of $\mathcal{F}_u$ near L. For $u \in \mathscr{D}(\mathcal{M}_0)$, the following complex function is defined by Yoshida [37]:

$$f(u) = \int_{s_u(\mathcal{M}_u \setminus L)} C - \frac{1}{2\pi} \int_{s_u(L)} (\theta_1 - i\theta_{23}), \tag{6-2}$$

where $s_u\colon \mathcal{M}_u \setminus L \to F(\mathcal{M}_u)$ and $s_u\colon L \to F(\mathcal{M}_u)$ are the sections defined by $\mathcal{F}_u$ and κ_u, respectively. By construction, the complex function $f(u)$ for $u \in \mathscr{D}(\mathcal{M}_0)$ extends the complex volume of $\mathcal{M}_0$ by $\mathbb{V}(\mathcal{M}_0) = \pi^2 f(0)$.

The following theorem was conjectured by Neumann and Zagier [26] and was proved by Yoshida [37]:

Theorem 6.2 (Yoshida [37]) *Over a neighborhood V of the origin in $\mathscr{D}(\mathcal{M}_0)$, the complex function $\exp(2\pi f)$ is holomorphic. If $u \in V$ represents the hyperbolic manifold $\mathcal{M}_u$ which can be completed to a closed hyperbolic manifold $\mathcal{M}_{p,q}$ by (p_i, q_i)–hyperbolic Dehn surgery to each end of $\mathcal{M}_u$, then*

$$\operatorname{Re} f(u) = \frac{1}{\pi^2} \operatorname{Vol}(\mathcal{M}_{p,q}) + \frac{1}{2\pi} \sum_{i=1}^{h} \operatorname{length}(\boldsymbol{g}_i),$$

$$\operatorname{Im} f(u) = 2\,\mathrm{CS}(\mathcal{M}_{p,q}) + \frac{1}{2\pi}\sum_{i=1}^{h} \mathrm{torsion}(\boldsymbol{g}_i) \mod \mathbb{Z},$$

where $\mathrm{length}(\boldsymbol{g}_i)$ *and* $\mathrm{torsion}(\boldsymbol{g}_i)$ *denote the length and the torsion of the closed geodesic* $\boldsymbol{g}_i$ *adjoined to the* i^{th} *end of* $\mathcal{M}_u$*, respectively.*

For the Zograf infinite products for complete hyperbolic manifolds with cusps $\mathcal{M}_0$, one can define these as in (5-13) and (5-17). Note that $F_n(s)$ and $G_n(s)$ as given in (3-17) and (3-18), respectively, are regular at $s=0$ under the given conditions on n for $\mathcal{M}_0$. This is because the statement in (3-2) is still true for a cofinite torsion-free subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$.

Moreover, one can consider the corresponding objects for $\mathcal{M}_u$ for u in a small open neighborhood V of the origin in $\mathscr{D}(\mathcal{M}_0)$ since the convergence condition is an open condition for u. Hence, for $u \in V$ we define

$$\begin{aligned} F_n(\mathcal{M}_u) &:= \prod_{[\gamma]_{\mathrm{p}}} \prod_{m=n}^{\infty} (1-\mathfrak{q}_\gamma^{m}) && \text{for } n \geq 3, \\ G_n(\mathcal{M}_u) &:= \prod_{[\gamma]_{\mathrm{p}}} \prod_{m=n}^{\infty} (1-\mathfrak{q}_\gamma^{m+1/2}) && \text{for } n \geq 2. \end{aligned} \tag{6-3}$$

Here the first product is taken over the set of conjugacy classes of the primitive loxodromic elements γ defined by ρ_u which are deformations of the loxodromic elements defined by ρ_0, and $\mathfrak{q}_\gamma$ is defined to be an eigenvalue of $\rho_u(\gamma)$ with modulus less than 1 for u in an open neighborhood V. Let us remark that there are loxodromic elements defined by ρ_u which are not deformations of loxodromic elements defined by ρ_0. Note that $F_n(\mathcal{M}_u)$ and $G_n(\mathcal{M}_u)$ are holomorphic functions over V since such elements are not included in the definition (6-3).

6.3 Spin structure

A spin structure on $\mathcal{M}_0$ naturally induces a spin structure on $\mathcal{M}_u$ for $u \in \mathscr{D}(\mathcal{M}_0)$. But, a spin structure on $\mathcal{M}_0$ can be extended to a spin structure on $\mathcal{M}_{p,q}$ with $p=(p_1,\dots,p_h)$ and $q=(q_1,\dots,q_h)$ obtained by (p_i,q_i)–hyperbolic Dehn surgery to each end of $\mathcal{M}_u$ only under a condition. By Proposition 5.2 in [18], a necessary and sufficient condition for this is that

$$\varepsilon_{m_i}^{p_i}\varepsilon_{l_i}^{q_i} = -1 \quad \text{for } i=1,\dots,h. \tag{6-4}$$

Here ε_{m_i} and ε_{l_i} denote the sign of the trace of an $\mathrm{SL}(2,\mathbb{C})$–lifting of $\rho_u(m_i)$ and $\rho_u(l_i)$, respectively. A spin structure on $\mathcal{M}_0$ is called *compactly approximable* if there are infinitely many $p=(p_1,\dots,p_h)$ and $q=(q_1,\dots,q_h)$ satisfying the conditions (6-1) and (6-4). In other words, a spin structure on $\mathcal{M}_0$ is compactly approximable if there is a sequence $\{\mathcal{M}_{p,q}\}$ of infinitely many spin closed hyperbolic manifolds such that the spin structure on $\mathcal{M}_{p,q}$ is induced from that of $\mathcal{M}_0$.

When $k=2n$, the representation ρ_u^{2n} does not depend on the choice of a spin structure on $\mathcal{M}_u$ since the representation ρ_u^{2n} factors through $\mathrm{PSL}(2,\mathbb{C})$. But, the homology groups $H_*(\mathcal{M}_u,\rho_u^{2n})$ need not vanish in general. The Reidemeister torsion $\mathcal{T}(\mathcal{M}_u,\rho_u^{2n})$ is an invariant of $(\mathcal{M}_u,\rho_u^{2n})$ valued in $\mathbb{C}^*/\{\pm1\}$ depending on the choice of basis of $H_*(\mathcal{M}_u,\rho_u^{2n})$. By Proposition 5.10 in [18], a collection $\{c_i\}$ of cycles in $H_1(T_i,\mathbb{Z})$ induces a basis of $H_*(\mathcal{M}_u,\rho_u^{2n})$. Hence, the Reidemeister torsion $\mathcal{T}(\mathcal{M}_u,\rho_u^{2n})$ can be considered as an invariant of $(\mathcal{M}_u,\rho_u^{2n},\{c_i\})$, which we denote by $\mathcal{T}(\mathcal{M}_u,\rho_u^{2n},\{c_i\})$.

When $k=2n-1$, by Corollary 5.3 in [18], a spin structure on $\mathcal{M}_0$ is compactly approximable if and only if it is *acyclic*, that is, $H_*(\mathcal{M}_0,\rho_0^{2n-1})=0$ for all $n\in\mathbb{N}$, where ρ_0^{2n-1} is defined by the chosen spin structure. Moreover, by the upper semicontinuous property of the dimension of $H_*(\mathcal{M}_u,\rho_u^{2n-1})$ (see [18, Section 5]), there exists an open neighborhood V of the origin in $\mathscr{D}(\mathcal{M}_0)$ such that $H_*(\mathcal{M}_u,\rho_u^{2n-1})=0$ for $u\in V$. Hence, the Reidemeister torsion $\mathcal{T}(\mathcal{M}_u,\rho_u^{2n-1})$ is a well-defined invariant valued in $\mathbb{C}^*/\{\pm1\}$ for $u\in V$ if a spin structure over $\mathcal{M}_0$ is acyclic.

7 The case of hyperbolic 3–manifolds with cusps and $\rho^{2(n-1)}$

For a given hyperbolic 3–manifold with cusps $\mathcal{M}_0$, we take a sequence $\{\mathcal{M}_{p,q}\}$ of infinitely many compact hyperbolic 3–manifolds, which are obtained by (p_i,q_i)–hyperbolic Dehn surgery to each end of $\mathcal{M}_u$ for points u near the origin in $\mathscr{D}(\mathcal{M}_0)$ satisfying (6-1).

Now, for a closed hyperbolic manifold $\mathcal{M}_{p,q}$, by Proposition 5.1 and Theorem 5.2, we have

$$|\mathcal{T}(\mathcal{M}_{p,q},\rho^{2(n-1)})|^{-1}=\Big|\exp\Big(\frac{1}{\pi}\big(n^2-n+\tfrac{1}{6}\big)\mathbb{V}(\mathcal{M}_{p,q})\Big)F_n(\mathcal{M}_{p,q})\Big|. \tag{7-1}$$

Here $\rho^{2(n-1)}$ is the representation of $\pi_1(\mathcal{M}_{p,q})$ to $\mathrm{SL}(S^{2(n-1)}(\mathbb{C}^2))$ which is defined in the same way as $\rho_u^{2(n-1)}$ for the corresponding point $u\in\mathscr{D}(\mathcal{M}_0)$.

By the Mayer–Vietoris argument for the Reidemeister torsion as in Lemma 3.12 of [18], or in [27, Section 3], we have

$$\mathcal{T}(\mathcal{M}_{p,q},\rho^{2(n-1)})=\mathcal{T}(\mathcal{M}_u,\rho^{2(n-1)},\{p_i m_i+q_i l_i\})\prod_{i=1}^{h}\prod_{m=1}^{n-1}(\mathfrak{q}_{\gamma_i}^{m}-1)(\mathfrak{q}_{\gamma_i}^{-m}-1), \tag{7-2}$$

where $\mathfrak{q}_{\gamma_i}=\exp(-(l_{\gamma_i}+i\theta_{\gamma_i}))$. Here γ_i for $i=1,\dots,h$ denotes the primitive loxodromic element corresponding to the added closed geodesic $\boldsymbol{g}_i$ to the i^{th} end of $\mathcal{M}_u$. Let us remark the equality (7-2) holds up to sign since the definition of Reidemeister torsion has ± 1 ambiguity.

For the complex volume of $\mathcal{M}_{p,q}$, by Theorem 6.2, we have

$$\exp\Bigl(\frac{2}{\pi}\mathbb{V}(\mathcal{M}_{p,q})\Bigr)=\exp(2\pi f(u))\prod_{i=1}^{h}\mathfrak{q}_{\gamma_i}. \tag{7-3}$$

For the Zograf infinite product $F_n(\mathcal{M}_{p,q})$, we separate the terms of γ_i for $i=1,\dots,h$ from other terms by

$$F_n(\mathcal{M}_{p,q})=\prod_{i=1}^{h}\prod_{m=n}^{\infty}(1-\mathfrak{q}_{\gamma_i}^{m})^2\prod_{[\gamma]_{\mathrm{p}}\neq[\gamma_i]}\prod_{m=n}^{\infty}(1-\mathfrak{q}_{\gamma}^{m}). \tag{7-4}$$

Note that the primitive conjugacy classes corresponding γ_i and γ_i^{-1} contribute by the same factor so that we get $(1-\mathfrak{q}_{\gamma_i}^{m})^2$. To deal with the second factor on the right-hand side of (7-4), we consider

$$\log\Bigl(\prod_{[\gamma]_{\mathrm{p}}\neq[\gamma_i]}\prod_{m=n}^{\infty}(1-\mathfrak{q}_{\gamma}^{m})\Bigr)=\int_{|z|>1}\sum_{m=n}^{\infty}(1-z^{-n})\,d\mu(\mathcal{M}_{p,q})(z). \tag{7-5}$$

Here the measure $\mu(\mathcal{M}_{p,q})$ on the complex plane is defined by

$$\mu(\mathcal{M}_{p,q})=\sum_{[\gamma]_{\mathrm{p}}\neq[\gamma_i]}\delta_{\mathfrak{q}_{\gamma}^{-1}}, \tag{7-6}$$

where the sum is taken over the set of conjugacy classes of the primitive loxodromic elements γ which are not the γ_i for $i=1,\dots,h$ in the image of the holonomy representation of $\pi_1(\mathcal{M}_{p,q})$. One can define the corresponding measure $\mu(\mathcal{M}_0)$, which is defined as in (7-6), where the sum is taken over the same set used in the definition of $F_n(\mathcal{M}_0)$. By Theorems 6.5 and 6.6 of [18], the measure $\mu(\mathcal{M}_{p,q})$ weakly converges to $\mu(\mathcal{M}_0)$ as $u=(u_1,\dots,u_h)$ goes to the origin of $\mathscr{D}(\mathcal{M}_0)$ satisfying the condition (6-1). From now on, the limit as $u \to 0$ should be understood in this sense. Hence, we have

$$\lim_{u\to 0}\Bigl(\prod_{[\gamma]_{\mathrm{p}}\neq[\gamma_i]}\prod_{m=n}^{\infty}(1-\mathfrak{q}_\gamma^m)\Bigr) = F_n(\mathcal{M}_0). \tag{7-7}$$

Combining the equalities (7-1), (7-2), (7-3) and (7-4), we have

$$\begin{aligned}\Bigl|\mathcal{T}(M_u,\rho_u^{2(n-1)},\{p_i m_i + q_i l_i\})\prod_{i=1}^{h}\mathfrak{q}_{\gamma_i}^{1/12}\prod_{m=1}^{\infty}(1-\mathfrak{q}_{\gamma_i}^m)^2\Bigr|^{-1}& \\ = \bigl|\exp\bigl((n^2-n+\tfrac{1}{6})\pi f(u)\bigr)(F_n(\mathcal{M}_0)+\varepsilon_1(u))\bigr|&,\end{aligned} \tag{7-8}$$

where $\varepsilon_1(u)\in\mathbb{C}$ is such that $\lim_{u\to 0}\varepsilon_1(u)=0$.

Let us recall the equality given in [26, Section 4],

$$l_{\gamma_i} + i\theta_{\gamma_i} = -(r_i u_i + s_i v_i) \mod 2\pi i,$$

where r_i and s_i are integers such that $p_i s_i - q_i r_i = 1$. By this and (6-1),

$$\widetilde{\tau}_i(u) := \frac{r_i + s_i\tau_i(u)}{p_i + q_i\tau_i(u)} = -\frac{1}{2\pi i}(l_{\gamma_i}+i\theta_{\gamma_i}) \mod \mathbb{Z}, \tag{7-9}$$

where $\tau_i(u)$ is as given in Theorem 6.1. From (7-9), we can see that:

(7-10) The action of $\bigl(\begin{smallmatrix} p & q\\ r & s\end{smallmatrix}\bigr)$ on $\bigl(\begin{smallmatrix} m_i\\ l_i\end{smallmatrix}\bigr)$ induces the action of $\bigl(\begin{smallmatrix} s & r\\ q & p\end{smallmatrix}\bigr)$ on $\bigl(\begin{smallmatrix} \tau\\ 1\end{smallmatrix}\bigr)$.

Note that there exists an open neighborhood V of the origin of $\mathscr{D}(\mathcal{M}_0)$ such that $\widetilde{\tau}_i(u)$ lies in the upper half-plane for $u\in V$. Hence we can consider the Dedekind eta function of $\widetilde{\tau}_i(u)$ for $u\in V$. Recall that the Dedekind eta function $\eta(\tau)$ for τ in the upper half-plane is defined by

$$\eta(\tau) = e^{2\pi i\tau/24}\prod_{m=1}^{\infty}(1-\exp(2\pi i m\tau)),$$

which satisfies the transformation law

$$\log\eta\Bigl(\frac{s\tau+r}{q\tau+p}\Bigr) = \log\eta(\tau) + \tfrac{1}{4}\log(-(q\tau+p)^2) + \tfrac{1}{12}\pi i I, \tag{7-11}$$

where I is an integer depending on $\bigl(\begin{smallmatrix} s & r\\ q & p\end{smallmatrix}\bigr)\in \mathrm{SL}(2,\mathbb{Z})$.

By the definition of the Reidemeister torsion, we have the equality

$$\mathcal{T}(\mathcal{M}_u,\rho_u^{2(n-1)},\{p_i m_i + q_i l_i\}) = \mathcal{T}(\mathcal{M}_u,\rho_u^{2(n-1)},\{m_i\})A_{2(n-1)}(u)^{-1}, \tag{7-12}$$

where $A_{2(n-1)}(u)$ denotes the determinant of the basis-changing matrix from the one determined by $\{m_i\}$ to the one determined by $\{p_i m_i + q_i l_i\}$ as explained in Section 6.3. Hence, using (7-12), the equality (7-8) can be rewritten in terms of the Dedekind eta function $\eta(\tilde{\tau}_i(u))$ as

$$\left|\mathcal{T}(\mathcal{M}_u, \rho_u^{2(n-1)}, \{m_i\}) A_{2(n-1)}(u)^{-1} \prod_{i=1}^{h} \eta(\tilde{\tau}_i(u))^2\right|^{-1} = \left|\exp\left(\left(n^2 - n + \tfrac{1}{6}\right)\pi f(u)\right)(F_n(\mathcal{M}_0) + \varepsilon_1(u))\right|. \tag{7-13}$$

As in the proof of Lemma 5.13 of [18], one can check that

$$\lim_{u\to 0}\left(A_{2(n-1)}(u)^{-1} \prod_{i=1}^{h}(q_i \tau_i(u) + p_i)\right) = 1. \tag{7-14}$$

This and the equality (7-11) imply

$$\lim_{u\to 0}\left|A_{2(n-1)}(u)^{-1} \prod_{i=1}^{h} \eta(\tilde{\tau}_i(u))^2\right| = \left|\prod_{i=1}^{h} \eta(\tau_i(0))^2\right|. \tag{7-15}$$

Hence, we have

$$\left|\mathcal{T}(\mathcal{M}_u, \rho_u^{2(n-1)}, \{m_i\}) \left(\prod_{i=1}^{h} \eta(\tau_i(0))^2 + \varepsilon_2(u)\right)\right|^{-1} = \left|\exp\left(\left(n^2 - n + \tfrac{1}{6}\right)\pi f(u)\right)(F_n(\mathcal{M}_0) + \varepsilon_1(u))\right|, \tag{7-16}$$

where $\varepsilon_2(u) \in \mathbb{C}$ such that $\lim_{u\to 0} \varepsilon_2(u) = 0$. Taking $u \to 0$ along the discrete set corresponding to the sequence $\{\mathcal{M}_{p,q}\}$, we obtain the corresponding equality for $\mathcal{M}_0$. This completes the proof of Theorem 1.1.

8 The case of hyperbolic 3–manifolds with cusps and $\rho^{2(n-1)}$

In this section we prove:

Theorem 8.1 *Let $\mathcal{M}_0$ be a complete hyperbolic 3–manifold of finite volume with h cusps. For an acyclic spin structure on $\mathcal{M}_0$, for $n \geq 2$,*

$$\left|\mathcal{T}(\mathcal{M}_0, \rho^{2n-1}) \prod_{i=1}^{h} (\theta_{01}(0, \tau_i)\eta(\tau_i)^{-1})\right|^{-1} = \left|\exp\left(\frac{1}{\pi}\left(n^2 - \tfrac{1}{12}\right)\mathbb{V}(\mathcal{M}_0)\right) G_n(\mathcal{M}_0)\right|. \tag{8-1}$$

Here $\theta_{01}(z,\tau)$ is a theta function defined by

$$\theta_{01}(z,\tau)=\sum_{n\in\mathbb{Z}}\exp\bigl(\pi i n^2\tau+2\pi i n\bigl(z+\tfrac{1}{2}\bigr)\bigr) \tag{8-2}$$

for $z\in\mathbb{C}$ and τ in the upper half-plane.

Proof Although we prove this theorem essentially in the same way as the proof of Theorem 1.1, we need to explain how the acyclic spin structure is involved in the following proof. If the given spin structure on $\mathcal{M}_0$ is acyclic, it is compactly approximable, as explained in Section 6.3. Then, for a basis (m_i,l_i) of $H_1(T_i,\mathbb{Z})$ for $i=1,\dots,h$, there are infinitely many points near the origin of $\mathscr{D}(\mathcal{M}_0)$ with coprime pairs (p_i,q_i) for $i=1,\dots,h$ satisfying (6-1) and (6-4). In particular, if necessary properly changing the basis (m_i,l_i) to have $\varepsilon_{m_i}=-1$ for $i=1,\dots,h$, we may assume that

$$p_i=4k_i+1,\quad q_i=4l_i\quad \text{for } k_i,l_i\in\mathbb{Z}. \tag{8-3}$$

For the closed hyperbolic manifold $\mathcal{M}_{p,q}$ with the induced spin structure, by Proposition 5.1 and Theorem 5.3 we have

$$|\mathcal{T}(\mathcal{M}_{p,q},\rho^{2n-1})|^{-1}=\Bigl|\exp\Bigl(\frac{1}{\pi}\bigl(n^2-\tfrac{1}{12}\bigr)\mathbb{V}(\mathcal{M}_{p,q})\Bigr)G_n(\mathcal{M}_{p,q})\Bigr|. \tag{8-4}$$

By the Mayer–Vietoris argument for the Reidemeister torsion as in Lemma 3.7 of [18], we have

$$\mathcal{T}(\mathcal{M}_{p,q},\rho^{2n-1})=\mathcal{T}(\mathcal{M}_u,\rho^{2n-1})\prod_{i=1}^{h}\prod_{m=0}^{n-1}(\mathfrak{q}_{\gamma_i}^{m+1/2}-1)(\mathfrak{q}_{\gamma_i}^{-(m+1/2)}-1), \tag{8-5}$$

where $\mathfrak{q}_{\gamma_i}^{j+1/2}=\exp\bigl(-\bigl(j+\tfrac{1}{2}\bigr)(l_{\gamma_i}+i\theta_{\gamma_i})\bigr)$ is defined with respect to the induced spin structure on $\mathcal{M}_{p,q}$.

For the Zograf infinite product $G_n(\mathcal{M}_{p,q})$, by Theorems 6.5 and 6.6 in [18] as before,

$$G_n(\mathcal{M}_{p,q})=\prod_{i=1}^{h}\prod_{m=n}^{\infty}(1-\mathfrak{q}_{\gamma_i}^{m+1/2})^2(G_n(\mathcal{M}_0)+\varepsilon_3(u)), \tag{8-6}$$

where $\varepsilon_3(u)\in\mathbb{C}$ such that $\lim_{u\to 0}\varepsilon_3(u)=0$. Here note that the primitive conjugacy classes corresponding to γ_i and γ_i^{-1} contribute by the same factor, so we get $(1-\mathfrak{q}_{\gamma_i}^{m+1/2})^2$.

Combining equalities (7-3), (8-4), (8-5) and (8-6),

$$\left|\mathcal{T}(\mathcal{M}_u,\rho_u^{2n-1})\prod_{i=1}^{h}\mathfrak{q}_{\gamma_i}^{-1/24}\prod_{m=0}^{\infty}(1-\mathfrak{q}_{\gamma_i}^{m+1/2})^2\right|^{-1} = \left|\exp\bigl(\bigl(n^2-\tfrac{1}{12}\bigr)\pi f(u)\bigr)(G_n(\mathcal{M}_0)+\varepsilon_3(u))\right|. \tag{8-7}$$

From the formula given on page 69 in [24], let us recall that the theta function $\theta_{01}(z,\tau)$ has a product expression at $z=0$,

$$\theta_{01}(0,\tau)=\prod_{m=1}^{\infty}(1-\exp(2\pi i m\tau))\prod_{m=0}^{\infty}\bigl(1-\exp(\pi i(2m+1)\tau)\bigr)^2. \tag{8-8}$$

By Proposition 9.2 in [24], it also satisfies the transformation law

$$\theta_{01}(0,\gamma\tau)^2=(q\tau+p)\theta_{01}(0,\tau)^2 \tag{8-9}$$

for $\gamma\in\Gamma(4)$. Here $\Gamma(4)\subset \mathrm{SL}(2,\mathbb{Z})$ denotes the principal congruence group of level 4, and the action $\gamma\tau$ is given by $\frac{s\tau+r}{q\tau+p}$ for an element $\gamma=\left(\begin{smallmatrix} s & r\\ q & p\end{smallmatrix}\right)$ in $\Gamma(4)$, recalling (7-10). Now, properly changing k_i and l_i in (8-3) if needed, one may assume

$$\gamma_i=\begin{pmatrix} s_i & r_i\\ q_i & p_i\end{pmatrix}\in\Gamma(4). \tag{8-10}$$

Then, for these γ_i for $i=1,\dots,h$, we have the equality

$$\begin{aligned}|\mathfrak{q}_{\gamma_i}|\prod_{m=0}^{\infty}|1-\mathfrak{q}_{\gamma_i}^{m+1/2}|^{-48} &= |\exp(2\pi i\tilde{\tau}_i(u))|\prod_{m=0}^{\infty}|1-\exp(\pi i(2m+1)\tilde{\tau}_i(u))|^{-48}\\ &= |\eta(\tilde{\tau}_i(u))|^{24}|\theta_{01}(0,\tilde{\tau}_i(u))|^{-24}=|\eta(\tau_i(u))|^{24}|\theta_{01}(0,\tau_i(u))|^{-24}.\end{aligned} \tag{8-11}$$

By (8-7) and (8-11), we have

$$\left|\mathcal{T}(\mathcal{M}_u,\rho_u^{2n-1})\prod_{i=1}^{h}\bigl(\theta_{01}(0,\tau_i(u))\eta(\tau_i(u))^{-1}\bigr)\right|^{-1} = \left|\exp\bigl(\bigl(n^2-\tfrac{1}{12}\bigr)\pi f(u)\bigr)(G_n(\mathcal{M}_0)+\varepsilon_3(u))\right|. \tag{8-12}$$

Taking $u\to 0$ along the discrete set corresponding to the sequence $\{\mathcal{M}_{p,q}\}$, we obtain the corresponding equality for $\mathcal{M}_0$ with acyclic spin structure. This completes the proof for the case of hyperbolic 3–manifolds with cusps and ρ^{2n-1}. □

By the same reasoning as before, Theorems 5.3 and 8.1 lead the author to make the following conjecture:

Conjecture 8.2 There exists an open neighborhood V of the origin in $\mathscr{D}(\mathcal{M}_0)$ where, for $n \geq 2$,

$$\mathcal{T}(\mathcal{M}_u, \rho_u^{2n-1})^{-24} \prod_{i=1}^{h} \bigl(\theta_{01}(0, \tau_i(u))\eta(\tau_i(u))^{-1}\bigr)^{-24} = c_{\mathcal{M}_0,n} \exp(2(12n^2-1)\pi f(u)) G_n(\mathcal{M}_u)^{24},$$

where $c_{\mathcal{M}_0,n}$ is a constant depending only on $\mathcal{M}_0$ and n with $|c_{\mathcal{M}_0,n}| = 1$.

Let us remark that we need to take the 24^{th} power of the equality (8-1) to have well-defined complex functions over $V \subset \mathscr{D}(\mathcal{M}_0)$.

References

[1] **L V Ahlfors**, *Complex analysis: an introduction to the theory of analytic functions of one complex variable*, 3rd edition, McGraw-Hill, New York (1978) MR

[2] **M F Atiyah**, **V K Patodi**, **I M Singer**, *Spectral asymmetry and Riemannian geometry, I*, Math. Proc. Cambridge Philos. Soc. 77 (1975) 43–69 MR

[3] **M F Atiyah**, **V K Patodi**, **I M Singer**, *Spectral asymmetry and Riemannian geometry, II*, Math. Proc. Cambridge Philos. Soc. 78 (1975) 405–432 MR

[4] **J-M Bismut**, **W Zhang**, *An extension of a theorem by Cheeger and Müller*, Astérisque 205, Soc. Math. France, Paris (1992) MR

[5] **A Borel**, **N Wallach**, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, 2nd edition, Mathematical Surveys and Monographs 67, Amer. Math. Soc., Providence, RI (2000) MR

[6] **U Bunke**, **M Olbrich**, *Selberg zeta and theta functions: a differential operator approach*, Mathematical Research 83, Akademie, Berlin (1995) MR

[7] **S E Cappell**, **E Y Miller**, *Complex-valued analytic torsion for flat bundles and for holomorphic bundles with* $(1,1)$ *connections*, Comm. Pure Appl. Math. 63 (2010) 133–202 MR

[8] **K Fedosova**, *The twisted Selberg trace formula and the Selberg zeta function for compact orbifolds*, preprint (2015) arXiv

[9] **P B Gilkey**, *Invariance theory, the heat equation, and the Atiyah–Singer index theorem*, 2nd edition, CRC, Boca Raton, FL (1995) MR

[10] **I C Gohberg**, **M G Kreĭn**, *Introduction to the theory of linear nonselfadjoint operators*, Translations of Mathematical Monographs 18, Amer. Math. Soc., Providence, RI (1969) MR

[11] **Y Gon**, **J Park**, *The zeta functions of Ruelle and Selberg for hyperbolic manifolds with cusps*, Math. Ann. 346 (2010) 719–767 MR

[12] **R Goodman**, **N R Wallach**, *Representations and invariants of the classical groups*, Encyclopedia of Mathematics and its Applications 68, Cambridge Univ. Press (1998) MR

[13] **L Hörmander**, *The analysis of linear partial differential operators, I: Distribution theory and Fourier analysis*, 2nd edition, Grundl. Math. Wissen. 256, Springer (1990) MR

[14] **A Marden**, *Outer circles: an introduction to hyperbolic* 3*–manifolds*, Cambridge Univ. Press (2007) MR

[15] **Y Matsushima**, **S Murakami**, *On vector bundle valued harmonic forms and automorphic forms on symmetric riemannian manifolds*, Ann. of Math. 78 (1963) 365–416 MR

[16] **A McIntyre**, **J Park**, *Tau function and Chern–Simons invariant*, Adv. Math. 262 (2014) 1–58 MR

[17] **A McIntyre**, **L A Takhtajan**, *Holomorphic factorization of determinants of Laplacians on Riemann surfaces and a higher genus generalization of Kronecker's first limit formula*, Geom. Funct. Anal. 16 (2006) 1291–1323 MR

[18] **P Menal-Ferrer**, **J Porti**, *Higher-dimensional Reidemeister torsion invariants for cusped hyperbolic* 3*–manifolds*, J. Topol. 7 (2014) 69–119 MR

[19] **J J Millson**, *Closed geodesics and the* η*–invariant*, Ann. of Math. 108 (1978) 1–39 MR

[20] **J Milnor**, *Whitehead torsion*, Bull. Amer. Math. Soc. 72 (1966) 358–426 MR

[21] **W Müller**, *Analytic torsion and* R*–torsion for unimodular representations*, J. Amer. Math. Soc. 6 (1993) 721–753 MR

[22] **W Müller**, *A Selberg trace formula for non-unitary twists*, Int. Math. Res. Not. 2011 (2011) 2068–2109 MR

[23] **W Müller**, *The asymptotics of the Ray–Singer analytic torsion of hyperbolic* 3*–manifolds*, from "Metric and differential geometry" (X Dai, X Rong, editors), Progr. Math. 297, Springer (2012) 317–352 MR

[24] **D Mumford**, *Tata lectures on theta, I*, Progr. Math. 28, Birkhäuser, Boston, MA (1983) MR

[25] **W D Neumann**, **J Yang**, *Bloch invariants of hyperbolic* 3*–manifolds*, Duke Math. J. 96 (1999) 29–59 MR

[26] **W D Neumann**, **D Zagier**, *Volumes of hyperbolic three-manifolds*, Topology 24 (1985) 307–332 MR

[27] **J Park**, *Half-density volumes of representation spaces of some* 3*–manifolds and their application*, Duke Math. J. 86 (1997) 493–515 MR

[28] **J Park**, *Analytic torsion and Ruelle zeta functions for hyperbolic manifolds with cusps*, J. Funct. Anal. 257 (2009) 1713–1758 MR

[29] **J Park**, *Reidemeister torsion, complex volume, and Zograf infinite product*, preprint (2016) arXiv

[30] **J Park**, *Reidemeister torsion, complex volume, and Zograf infinite product for hyperbolic* 3*–manifolds with cusps*, preprint (2017) arXiv

[31] **M A Shubin**, *Pseudodifferential operators and spectral theory*, 2nd edition, Springer (2001) MR

[32] **P Spilioti**, *Selberg and Ruelle zeta functions for non-unitary twists*, Ann. Global Anal. Geom. 53 (2018) 151–203 MR

[33] **M E Taylor**, *Pseudodifferential operators*, Princeton Mathematical Series 34, Princeton Univ. Press (1981) MR

[34] **W P Thurston**, *Three-dimensional geometry and topology, I*, Princeton Mathematical Series 35, Princeton Univ. Press (1997) MR

[35] **N R Wallach**, *On the Selberg trace formula in the case of compact quotient*, Bull. Amer. Math. Soc. 82 (1976) 171–195 MR

[36] **A Wotzke**, *Die Ruellesche Zetafunktion und die analytische Torsion hyperbolischer Mannigfaltigkeiten*, PhD thesis, Universtät Bonn (2008) Available at `https://bib.math.uni-bonn.de/downloads/bms/BMS-389.pdf`

[37] **T Yoshida**, *The* η*–invariant of hyperbolic* 3*–manifolds*, Invent. Math. 81 (1985) 473–514 MR

[38] **C K Zickert**, *The volume and Chern–Simons invariant of a representation*, Duke Math. J. 150 (2009) 489–532 MR

[39] **P G Zograf**, *Liouville action on moduli spaces and uniformization of degenerate Riemann surfaces*, Algebra i Analiz 1 (1989) 136–160 MR In Russian; translated in Leningrad Math. J. 1 (1990) 941–965

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