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HALF-DENSITY VOLUMES OF REPRESENTATION SPACES OF SOME 3-MANIFOLDS AND THEIR APPLICATION

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§1. Introduction. In this paper we compute half-density volumes of the irreducible $SU(2)$-representation spaces of Seifert fibred manifolds and graph manifolds. The half-density over the irreducible $SU(2)$-representation space of a 3-manifold comes from the Reidemeister torsion for Ad-$SU(2)$-representation. More precisely, the determinant term of the first homology of the Reidemeister torsion gives the half-density of the irreducible representation space. This is because the tangent space of the irreducible representation space can be identified with the first cohomology of the twisted cochain complex.

The motivation of this paper is given by two sources. The first source is E. Witten's method to compute the symplectic volume of the irreducible $SU(2)$-representation space of a Riemann surface. In [W2] Witten suggested a useful method to compute the symplectic volume of this space using the Reidemeister torsion and the character theory of the Lie group $SU(2)$.

The second source is the invariant defined by L. C. Jeffrey and J. Weitsman in [JW1]. This invariant is motivated by the asymptotic expansion of the Witten invariant of a 3-manifold. Jeffrey and Weitsman define this invariant using the Reidemeister torsion as a half-density measure of the irreducible $SU(2)$-representation space. Hence, to compute this invariant we must compute the Reidemeister torsion completely, including the determinant term of the homology, which is the half-density measure of the $SU(2)$-representation space. In this paper we call this invariant the Jeffrey-Weitsman-Witten invariant.

From the above motivations, we could consider naturally that this invariant might be computed by the method of [W2]. The examples to which we apply the method of Witten are Seifert fibred manifolds and graph manifolds. This is because these manifolds are made from the trivial circle bundle over the Riemann surfaces by twisting finite fibres. So the method of Witten is applicable with some modification.

To compute the half-density derived from the Reidemeister torsion, we must compute both the scalar part and the determinant part of the Reidemeister torsion. The method to compute the scalar part comes from [F]. For Seifert fibred manifolds and graph manifolds, this value gives the weight of the half-density to each connected component of the irreducible $SU(2)$-representation space. So we combine two methods of [F] and [W2] to compute the half-density volumes of

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the irreducible $SU(2)$-representation spaces for Seifert fibred manifolds and graph manifolds.

Our computing method is "to cut and paste" with a topological viewpoint. This can be comparable to the method of P. Kirk and E. Klassen to compute the Chern-Simons invariants for 3-manifolds [KK]. We decompose the given manifolds into simple pieces with which we can deal easily, and then we glue the data of the decomposed pieces and investigate the gluing maps. The data of the pieces and the gluing maps gives the result that we want.

Now we explain how this paper is organized. In Section 2 we study basic examples, which are the building blocks of Seifert fibred manifolds and graph manifolds. We study the Reidemeister torsions and the $SU(2)$-representation spaces of these basic examples. In Section 3 we compute the scalar part of the Reidemeister torsions of Seifert fibred manifolds and graph manifolds for Ad-$SU(2)$-representations. The computing method of Section 3 comes from [F]. This method is exactly "to cut and paste" so that we can apply the result of Section 2. In Section 4 we integrate the determinant term of the first homology part of the Reidemeister torsion over the irreducible $SU(2)$-representation space. The integration process of this section is also "to cut and paste." We get the half-density volume of the irreducible $SU(2)$-representation space by investigation of the pasting process for the half densities of the representation spaces of the decomposed pieces. In Section 5 we apply the result from Section 4 to compute the Jeffrey-Weitsman-Witten invariant for a Seifert fibred manifold whose irreducible $SU(2)$-representation space is nondiscrete. This gives the exact value of the Jeffrey-Weitsman-Witten invariant for a Seifert fibred manifold with the nondiscrete $SU(2)$-representation space combined with the result of D. R. Auckly for the Chern-Simons invariant for this manifold.

§2. Basic examples. In this section we study some basic examples, which we will use in Sections 3 and 4. We study the Reidemeister torsions and $SU(2)$-representation spaces of S^1, T^2, and pants P. These examples are the building blocks of some 3-manifolds with which we will deal later. From now on, R-torsion always means the Reidemeister torsion. For the definition and the basic property of the R-torsion, see [F].

Our first example is the circle S^1. Every $SU(2)$-representation of $\pi_1(S^1)$ is reducible, since $\pi_1(S^1)$ is abelian. Hence the representation is determined by the holonomy parameter u so that the $SU(2)$-representation ρ_u has the following form up to conjugation

$$\rho_u(1) = \begin{pmatrix} e^{2\pi i u} & 0 \\ 0 & e^{-2\pi i u} \end{pmatrix}$$

for the generator $1 \in \pi_1(S^1) = Z$. Hence the $SU(2)$-representation space for $\pi_1(S^1)$ is S^1, which can be identified with the maximal torus T^1 of $SU(2)$. We denote this space by $\mathscr{L}$. This representation space will be used importantly later. The R-torsion of S^1 for Ad-$SU(2)$-representation is denoted by $\tau(S^1, \mathrm{Ad}(\rho_u))$. R-torsion is given by the torsion of the following chain complex $C.(S^1, \mathrm{Ad}(\rho_u))$:

$$0 \to C_1(R^1) \otimes_\pi su(2) \to C_0(R^1) \otimes_\pi su(2) \to 0,$$

where R^1 is the universal covering space of S^1, and the tensor product is taken over the $\pi = \pi_1(S^1)$. Then the homology of $C.(S^1, \mathrm{Ad}(\rho_u))$ is

$$H_0(S^1, su(2)_\rho) = R[e \otimes v],$$

$$H_1(S^1, su(2)_\rho) = R[x \otimes v],$$

where R is the real number field, e, x are the 0,1-cells of S^1, respectively, and v is the Ad(ρ)-invariant vector in $su(2)$. Then we have that

$$(2.1) \qquad \tau(S^1, \mathrm{Ad}(\rho_u)) = 4\sin^2(2\pi u) D_{H_*(S^1)}^{-1} \in \det H_*(S^1, su(2)_\rho)^{-1},$$

where $D_{H_*(S^1)} = (e \otimes v) \otimes (x \otimes v)^{-1}$ and $\det H_*(S^1, su(2)_\rho) = \det H_0(S^1, su(2)_\rho) \otimes \det H_1(S^1, su(2)_\rho)^{-1}$. For the notation about the determinant line, we follow the notation of [F].

The next example is the torus T^2. Since $\pi_1(T^2)$ is also abelian, every $SU(2)$-representation of $\pi_1(T^2)$ is reducible with the following form:

$$\rho_{\alpha,\beta}((1,0)) = \begin{pmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{-2\pi i\alpha} \end{pmatrix}, \qquad \rho_{\alpha,\beta}((0,1)) = \begin{pmatrix} e^{2\pi i\beta} & 0 \\ 0 & e^{-2\pi i\beta} \end{pmatrix}$$

for a basis (1,0), (0,1) of $\pi_1(T^2)$. We know that $SU(2)$-representation space $R(T^2, SU(2))$ is an object called the "pillowcase." The R-torsion $\tau(T^2, \mathrm{Ad}(\rho_{\alpha,\beta}))$ is the torsion of the following chain complex $C.(T^2, \mathrm{Ad}(\rho_{\alpha,\beta}))$:

$$0 \to C_2(R^2) \otimes_\pi su(2) \to C_1(R^2) \otimes_\pi su(2) \to C_0(R^2) \otimes_\pi su(2) \to 0.$$

Then we have

$$H_0(T^2, su(2)_\rho) = R[e \otimes v],$$

$$H_1(T^2, su(2)_\rho) = R[x \otimes v, y \otimes v],$$

$$H_2(T^2, su(2)_\rho) = R[x \cup y \otimes v],$$

where R is the real number field, v is the Ad(ρ)-invariant vector in $su(2)$, and

$e, x, y, x \cup y$ are the 0,1,2 cells of T^2, respectively. From the Poincaré duality,

$$\tau(T^2, \mathrm{Ad}(\rho)) = (e \otimes v)^{-1} \otimes (x \otimes v \wedge y \otimes v) \otimes (x \cup y \otimes v)^{-1} \tag{2.2}$$
$$= D^{-1}_{H_*(T^2)} \in \det H_*(T^2, su(2)_\rho)^{-1},$$

where $\det H_*(T^2, su(2)_\rho) = \bigotimes_{i=0}^{2} H_i(T^2, su(2)_\rho)^{(-1)^i}$.

Our third example is the pants P. Contrary to above examples with abelian fundamental groups, $\pi_1(P)$ is the free group with two generators. Hence there are irreducible $SU(2)$-representations of $\pi_1(P)$. We can see that the $SU(2)$-representation space $R(P, SU(2))$ is the quotient space of $SU(2) \times SU(2)$ by $SO(3)$, since P is homotopy equivalent to the figure-eight simplex P'. We denote the irreducible $SU(2)$-representation space of $\pi_1(P)$ by $R(P, SU(2))^- \subset R(P, SU(2))$. Since P is homotopy equivalent to P', we consider a chain complex $C.(P', \mathrm{Ad}(\rho))$ instead of the chain complex $C.(P, \mathrm{Ad}(\rho))$ for an irreducible $SU(2)$-representation ρ

$$0 \to C_1(\tilde{P}') \otimes_\pi su(2) \to C_0(\tilde{P}') \otimes_\pi su(2) \to 0,$$

where $\tilde{P}'$ is the universal covering space of P'. Then we have

$$H_0(P, su(2)_\rho) = H_2(P, su(2)_\rho) = 0,$$
$$H_1(P, su(2)_\rho) = R[x_1 \otimes v_1, x_2 \otimes v_2, x_3 \otimes v_3],$$

where R is the real number field, x_i's are the three boundaries of P, and v_i's are the $\mathrm{Ad}(\rho(x_i))$-invariant vectors in $su(2)$. The R-torsion $\tau(P, \mathrm{Ad}(\rho))$ is the scalar multiple of

$$(x_1 \otimes v_1) \wedge (x_2 \otimes v_2) \wedge (x_3 \otimes v_3) \in \det H_*(P, su(2)_\rho)^{-1} \tag{2.3}$$

where $\det H_*(P, su(2)_\rho) = \det H_1(P, su(2)_\rho)$ because the representation ρ is irreducible.

In fact $\tau(P, \mathrm{Ad}(\rho))$ is a volume form of $R(P, SU(2))^-$. Now we consider more closely the volume form $\tau(P, \mathrm{Ad}(\rho))$. Let

$$\sigma\colon R(P, SU(2))^- \to \mathscr{L}_1 \times \mathscr{L}_2 \times \mathscr{L}_3$$

be the map induced by the restriction from P to $\partial P = \{x_1, x_2, x_3\}$, where $\mathscr{L}_i$ is the $SU(2)$-representation space for a circle x_i defined the same way as $\mathscr{L}$ for S^1. Recall that $\mathscr{L}_i$ can be identified with the maximal torus T^1 of $SU(2)$. By the definition, σ takes the conjugacy class of an irreducible representation ρ to the holonomies of $\rho(x_1), \rho(x_2), \rho(x_3)$. Since σ is injective map, there exists an object on

$\mathscr{L}_1 \times \mathscr{L}_2 \times \mathscr{L}_3$ whose pullback is the volume form $\tau(P, \mathrm{Ad}(\rho))$. We denote this by $\sigma_*\tau(P, \mathrm{Ad}(\rho))$. Since we can identify $\mathscr{L}_i$ with the maximal torus T^1, we have that

$$\sigma_*\tau(P, \mathrm{Ad}(\rho)) = f v_1 v_2 v_3$$

for some $f \in L^2(\mathscr{L}_1 \times \mathscr{L}_2 \times \mathscr{L}_3)$, where v_i is a natural volume form on the maximal torus T^1 with $\int_{T^1} v_i = 1$. From now on, $\mathrm{Vol}(G)$ is the volume of the compact Lie group G. Then we have the formula

$$\sigma_*\tau(P, \mathrm{Ad}(\rho)) = \frac{2}{\mathrm{Vol}(SU(2))^2} \sum_\alpha \frac{1}{n_\alpha} \prod_{i=1}^{3} \chi_\alpha(\rho(x_i)) v_1 v_2 v_3, \tag{2.4}$$

where the above sum is taken over all the irreducible representations of $SU(2)$, and n_α is the dimension of the representation space of an irreducible representation α. Note that the equality in (2.4) holds in L^2-sense. The proof of (2.4) is given in [W2].

If we assume that one boundary x_3 of P has a fixed holonomy, we must modify (2.4). The following formula for a boundary with the fixed holonomy is also used in [W2] without the explicit derivation. So we derive it here. In the following proposition, we use the character theory of $SU(2)$. For the details, see [BtD].

PROPOSITION 2.5. *If the trace of $\rho(x_3)$ is fixed so that $tr(\rho(x_3)) = 2\cos(\theta)$ for some fixed θ, then $\sigma_*\tau(P, \mathrm{Ad}(\rho))$ is*

$$\frac{2\,\mathrm{Vol}(S^2)}{\mathrm{Vol}(SU(2))^2} \sum_\alpha \frac{1}{n_\alpha} \frac{\sin(n_\alpha \theta)}{\sin(\theta)} \chi_\alpha(\rho(x_1)) \chi_\alpha(\rho(x_2)) v_1 v_2,$$

where the above sum is taken over all the irreducible representations of $SU(2)$ and n_α is the dimension of the representation space of an irreducible representation α. $\mathrm{Vol}(S^2)$ is the volume of S^2 induced from a volume form of $SU(2)$.

Proof. We can consider the set of conjugacy classes of representations satisfying the given condition as the inverse image of the map

$$p_3 \circ \sigma \colon R(P, SU(2))^- \to \mathscr{L}_1 \times \mathscr{L}_2 \times \mathscr{L}_3 \to \mathscr{L}_3$$

for $e^{(2\pi i \theta)} \in \mathscr{L}_3$, where p_3 is the natural projection. We denote this inverse image by $R(P, SU(2), \theta)^-$. And let $S^2(\theta)$ be the subset of $SU(2)$ with fixed trace $2\cos(\theta)$. Then there exists the natural projection map $p : SU(2) \times S^2(\theta) \to R(P, SU(2), \theta)^-$ with a fibre $SU(2)/Z_2$. As above, we have a map

$$\sigma' \colon R(P, SU(2), \theta)^- \to \mathscr{L}_1 \times \mathscr{L}_2,$$

which sends $[\rho]$ in $R(P, SU(2), \theta)^-$ to the holonomies $\rho(x_1)$, $\rho(x_2)$.

We push $\tau(P, \mathrm{Ad}(\rho))$ by σ' so that we get a volume form $\sigma'_*\tau(P, \mathrm{Ad}(\rho))$ on $\mathscr{L}_1 \times \mathscr{L}_2$. Then this volume can be written by $f v_1 v_2$ for some $f \in L^2(\mathscr{L}_1 \times \mathscr{L}_2)$. To find f exactly, we integrate the character χ_α of $\rho(x_1), \rho(x_2)$ for the irreducible representation α of $SU(2)$. This is because the set of characters $\{\chi_\alpha\}$ for all the irreducible representations of $SU(2)$ is the uniformly dense subset of the continuous function space of $\mathscr{L}_1 \times \mathscr{L}_2$. So we have the following equalities:

$$\begin{aligned}
W_{\alpha_1,\alpha_2} &= \int_{\mathscr{L}_1 \times \mathscr{L}_2} \chi_{\alpha_1}(\rho(x_1))\chi_{\alpha_2}(\rho(x_2))\sigma'_*\tau(P, \mathrm{Ad}(\rho)) \\
&= \frac{2}{\mathrm{Vol}(SU(2))} \int_{SU(2)\times S^2(\theta)} p^*\sigma'^*(\chi_{\alpha_1}(\rho(x_1))\chi_{\alpha_2}(\rho(x_2)))\, dU_2\, dg(\theta) \\
&= \frac{2}{\mathrm{Vol}(SU(2))} \int_{SU(2)\times S^2(\theta)} \chi_{\alpha_1}(g(\theta)^{-1}\rho(x_2)^{-1})\chi_{\alpha_2}(\rho(x_2))\, dU_2\, dg(\theta) \\
&= 2\frac{\delta_{\alpha_1,\alpha_2}}{n_{\alpha_1}} \int_{S^2(\theta)} \chi_{\alpha_1}(g(\theta)^{-1})\, dg(\theta) \\
&= 2\delta_{\alpha_1,\alpha_2} \frac{\chi_{\alpha_1}(g(\theta)^{-1})}{n_{\alpha_1}} \mathrm{Vol}(S^2(\theta)).
\end{aligned}$$

We have that

$$\sigma'_*\tau(P, \mathrm{Ad}(\rho)) = \frac{2\,\mathrm{Vol}(S^2(\theta))}{\mathrm{Vol}(SU(2))^2} \sum_\alpha \frac{1}{n_\alpha} \chi_\alpha(g(\theta)^{-1})\chi_\alpha(\rho(x_1))\chi_\alpha(\rho(x_2)) v_1 v_2.$$

On the other hand, $\chi_\alpha(g(\theta)^{-1}) = \sin(n_\alpha\theta)/\sin(\theta)$ and $\mathrm{Vol}(S^2(\theta)) = \mathrm{Vol}(S^2)$. So we get the formula in the proposition. □

§3. R-torsion of Seifert fibred manifolds and graph manifolds. In this section we compute the scalar part of the R-torsion of Seifert fibred manifolds and graph manifolds for the Ad-$SU(2)$-representation. This quantity gives the weight of the half-density for each connected component of the irreducible $SU(2)$-representation space of Seifert fibred manifold or graph manifold. Our computing method in this section comes from [F].

First we consider the manifolds whose R-torsion we will compute. We denote the Seifert fibred manifold with the Seifert invariant $\{g; (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)\}$ by $M = M(g; (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m))$. We have that $\pi_1(M)$ is

$$\left\{ a_i, b_i, q_j, h \colon [h, a_i] = [h, b_i] = [h, q_j] = 1, q_j^{\alpha_j} h^{\beta_j} = 1, \prod_{j=1}^m q_j \prod_{i=1}^g [a_i, b_i] = 1 \right\}.$$

We assume that $g \geqq 2$ from now on. The irreducible $SU(2)$-representation of $\pi_1(M)$ is well known. (See [A], [FS], [KK].) We review some facts that we need.

Since h is central in $\pi_1(M)$, an irreducible representation ρ takes h to ± 1 in $SU(2)$. So the trace of $\rho(q_j)$ is $2\cos(\pi n_j/\alpha_j)$. The set of numbers $\{n_1, \dots, n_m\}$, which are called "rotation numbers," determines a connected component of the irreducible $SU(2)$-representation space $R(M, SU(2))^-$. The rotation number n_j is even (odd) if β_j is even (odd).

Next we consider a graph manifold, which is made from two Seifert fibred manifolds M_1, M_2, which have a torus boundary each. We review some materials about graph manifolds of [KK]. M_1 is given by deleting a solid torus $D^2 \times S^1$ from M, where a disk D^2 lies in the base surface Σ_g and a circle S^1 is the fibre component of the Seifert fibration. Then $\pi_1(M_1)$ is

$$\pi_1(M_1) = \{a_i, b_i, q_j, h\colon [h, a_i] = [h, b_i] = [h, q_j] = 1, q_j^{\alpha_j} h^{\beta_j} = 1\}.$$

The other manifold M_2 can be constructed in the same way from another Seifert manifold $M(g'; (\alpha'_1, \beta'_1), \dots, (\alpha'_n, \beta'_n))$. Then $\pi_1(M_2)$ is

$$\pi_1(M_2) = \{a'_{i'}, b'_{i'}, r_{j'}, k\colon [k, a'_{i'}] = [k, b'_{i'}] = [k, r_{j'}] = 1, r_{j'}^{\alpha'_{j'}} k^{\beta'_{j'}} = 1\}.$$

Since $\partial M_1 = \partial M_2 = T^2$, we can glue two manifolds M_1, M_2 by an automorphism ϕ of T^2. We assume that the meridian, the longitude pairs of $\partial M_1, \partial M_2$, is given by

$$\{\mu_1, \lambda_1\} = \left\{\prod_{j=1}^{m} q_j \prod_{i=1}^{g} [a_i, b_i], h\right\},$$

$$\{\mu_2, \lambda_2\} = \left\{\prod_{j'=1}^{n} r_{j'} \prod_{i'=1}^{g'} [a'_{i'}, b'_{i'}], k\right\}.$$

Define $\phi\colon \partial M_1 \to \partial M_2$ by

$$\phi(\mu_1) = \alpha\mu_2 + \beta\lambda_2, \qquad (3.1)$$

$$\phi(\lambda_1) = \gamma\mu_2 + \delta\lambda_2$$

where $\alpha\delta - \beta\gamma = -1$. Then we have a glued manifold $N = M_1 \cup_\phi M_2$, and we call this manifold a graph manifold. Later we need to distinguish two cases, when $\gamma = 0$ and $\gamma \neq 0$. When we need to clarify the dependence of γ, we denote the graph manifold by N_γ to express the dependence of γ. The natural question is how we glue the irreducible $SU(2)$-representations ρ_1, ρ_2 of the manifolds M_1, M_2 to get an irreducible $SU(2)$-representation of graph manifold N. In fact, there is a condition of gluing representations. Since $\pi_1(\partial M_i)$ is abelian, we have

$$\rho_1(\mu_1) = \begin{pmatrix} e^{2\pi i\phi_1} & 0 \\ 0 & e^{-2\pi i\phi_1} \end{pmatrix}, \qquad \rho_1(\lambda_1) = \begin{pmatrix} e^{2\pi i\psi_1} & 0 \\ 0 & e^{-2\pi i\psi_1} \end{pmatrix},$$

$$\rho_2(\mu_2) = \begin{pmatrix} e^{2\pi i\phi_2} & 0 \\ 0 & e^{-2\pi i\phi_2} \end{pmatrix}, \qquad \rho_2(\lambda_2) = \begin{pmatrix} e^{2\pi i\psi_2} & 0 \\ 0 & e^{-2\pi i\psi_2} \end{pmatrix}.$$

We can see that $\psi_1, \psi_2 \in Z[1/2]$ since h, k are central. Then the condition that a glued representation ρ exists is that

$$\phi_1 = \alpha\phi_2 + \beta\psi_2, \qquad \psi_1 = \gamma\phi_2 + \delta\psi_2. \tag{3.2}$$

This condition comes from (3.1).

To compute the R-torsions of Seifert fibred manifolds and graph manifolds, we decompose these manifolds into simple pieces, which we considered in Section 2. For the Seifert fibred manifold $M(g; (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m))$, we can decompose M into

$$m\text{-solid tori } A_j = D_j^2 \times S^1 \quad \text{and} \quad (2(g-1)+m)\text{-copies of } X_i = P_i \times S^1,$$

where a disk D_j^2 lies on the base surface Σ_g, S^1 is the fibre component, and P_i's are the pairs of pants. The decomposition is the inverse process of construction of M and the pants decomposition of $\Sigma_g - \bigcup_{j=1}^m D_j^2$. We can assume that each X_i meets only one A_j or does not meet A_j by the isotopic moves of the exceptional fibres.

For graph manifolds $N = M_1 \cup_\phi M_2$, we decompose N into M_1, M_2, and then apply the same process to M_i as above. So we have

$$m\text{-solid tori } A_j,\ n\text{-solid tori } B_{j'},$$

$$\text{and} \quad (2g+m-1)\text{-copies of } X_i, (2g'+n-1)\text{-copies of } Y_{i'}$$

where X_i, $Y_{i'}$ are homeomorphic to $P \times S^1$ and

$$M_1 = \bigcup_{i=1}^{2g+m-1} X_i \bigcup_{j=1}^{m} A_j \quad \text{and} \quad M_2 = \bigcup_{i'=1}^{2g'+n-1} Y_{i'} \bigcup_{j'=1}^{n} B_{j'}.$$

We can assume that X_1 meets M_2, Y_1 meets M_1, and X_i ($Y_{i'}$) meets only one A_j ($B_{j'}$) or does not meet A_j ($B_{j'}$) as above.

We consider $SU(2)$-representations of $\pi_1(M)$ or $\pi_1(N)$ such that the induced representations by restriction to each piece X_i or X_i, $Y_{i'}$ are irreducible. We call such a representation a *totally irreducible representation* for M or N with respect to the above fixed decomposition.

As in the previous section, we denote the highest wedge product of a basis of $H_i(X, su(2)_{\rho_X})$ by $D_{H_i(X,su(2)_{\rho_X})}$ and $\bigotimes_{i=0}^{d}(D_{H_i(X,su(2)_{\rho_X})})^{(-1)^i}$ by $D_{H_*(X)}$ for a manifold X of d-dimension. We denote the dimension of $H_i(X, su(2)_{\rho_X})$ by $h_i(X)$ in the following proposition.

PROPOSITION 3.3. *Let ρ_M, ρ_{N_γ} be irreducible $SU(2)$-representations of $\pi_1(M)$, $\pi_1(N_\gamma)$ with rotation numbers $\{n_j\}, \{n_j, n'_{j'}\}$. We assume that ρ_M, ρ_{N_γ} are totally irreducible for M, N_γ. The R-torsion of the Seifert fibred manifold $M(g; (\alpha_j, \beta_j))$ and the graph manifold $N_\gamma = M_1(g; (\alpha_j, \beta_j)) \cup_{\phi_\gamma} M_2(g'; (\alpha'_{j'}, \beta'_{j'}))$ is given by*

$$\tau(M, \mathrm{Ad}(\rho_M)) = 4^m \prod_{j=1}^{m} \frac{\sin^2\left(\frac{\pi n_j \beta_j^*}{\alpha_j}\right)}{\alpha_j} D_{H_*(M)}^{-1} \in \det H_*(M, su(2)_{\rho_M})^{-1},$$

$$\tau(N_\gamma, \mathrm{Ad}(\rho_{N_\gamma})) = 4^{m+n} f(\gamma) \prod_{j=1}^{m} \frac{\sin^2\left(\frac{\pi n_j \beta_j^*}{\alpha_j}\right)}{\alpha_j} \prod_{j'=1}^{n} \frac{\sin^2\left(\frac{\pi n'_{j'} \beta'_{j}{}^*}{\alpha'_{j'}}\right)}{\alpha'_{j'}} D_{H_*(N_\gamma)}^{-1}$$

$$\in \det H_*(N_\gamma, su(2)_{\rho_N})^{-1}$$

where

$$\beta_j \beta_j^* = 1 \bmod \alpha_j, \quad \beta'_{j'} \beta'_{j'}{}^* = 1 \bmod \alpha'_{j'},$$

$$f(\gamma) = \begin{cases} 1 & \text{if } \gamma = 0 \\ \frac{1}{|\gamma|} & \text{if } \gamma \neq 0, \end{cases}$$

$h_1(M) = h_2(M)$ is $6(g-1) + 2m$ and $h_1(N_\gamma) = h_2(N_\gamma)$ is $6(g + g' - 1) + 2(m+n)$, $6(g+g') + 2(m+n) - 7$ if $\gamma = 0, \gamma \neq 0$.

Before we prove the proposition, we remark on some facts. We can see that the R-torsion depends on the gluing torus automorphisms of the decomposed pieces by $(\alpha_j, \beta_j), (\alpha'_{j'}, \beta'_{j'})$ and the representation ρ_M, ρ_N by rotation numbers $(n_j), (n_j, n'_{j'})$. Since Ad-$SU(2)$-representation is unimodular, the highest wedge product of $D_{H_i(M,su(2)_M)}$ does not depend on the basis change of $C_i(\tilde{M})$ when a basis of $su(2)$ is fixed. In the above proposition, we have that $D_{H_*(\cdot)}^{-1} = D_{H_1(\cdot, su(2))} \otimes D_{H_2(\cdot, su(2))}^{-1}$, since we assume that the representations are irreducible.

Proof. We shall compute the R-torsion of graph manifold N_γ for the totally irreducible $SU(2)$-representation with respect to the given decomposition. For Seifert fibred manifolds, we get the result in the process of computation of graph manifold.

First we define some tori from the decomposed pieces such that

$$T_j = \partial A_j, \qquad T'_{j'} = \partial B_{j'},$$

$$T_{i_1 i_2} = X_{i_1} \cap X_{i_2}, \qquad T'_{i'_1 i'_2} = Y_{i'_1} \cap Y_{i'_2}, \qquad T = M_1 \cap M_2.$$

We have the following formula about the R-torsions of each piece and the given manifold N_γ such that

$$\tau(N_\gamma) = \bigotimes_{\{i,i',j,j'\}} \tau(X_i) \cdot \tau(A_j) \cdot \tau(Y_{i'}) \cdot \tau(B_{j'}) \cdot \left(\bigotimes_{\{i,i',j,j'\}} \tau(T_j) \cdot \tau(T'_{j'}) \cdot \tau(T_{i_1 i_2}) \cdot \tau(T'_{i'_1 i'_2}) \cdot \tau(T) \right)^{-1} \bigotimes_{r=1}^{\infty} \tau(E^r_\cdot)^{-1}, \tag{3.4}$$

where $\cdot$ denotes the tensor product and the last term $\tau(E^r_\cdot)$ is the torsion of the spectral sequence of the generalized Meyer-Vietoris sequence. For the proof of (3.4), see [F]. In the above formula, we omit the representation notation for convenience.

Since $A_j, B_{j'}$ are homotopy equivalent to S^1, we can consider S^1 instead of $A_j, B_{j'}$ for the R-torsion. Let e be the 0-cell of each piece, and $a_j, b_{j'}$ be 1-cells of S^1's which are homotopy equivalent to $A_j, B_{j'}$. Then we have that $D_{H_*(A_j)} = e \otimes a_j^{-1}$ and $D_{H_*(B_{j'})} = e \otimes b_{j'}^{-1}$. We omit the notation of invariant vectors of $\mathrm{Ad}(\rho)(a_j), \mathrm{Ad}(\rho)(b_{j'})$ for convenience. Then we have the following formulas from the construction of N_γ and (2.1):

$$\begin{aligned} \tau(A_j) &= 4\sin^2\left(\frac{\pi n_j \gamma_j}{\alpha_j}\right) D^{-1}_{H_*(A_j)} \\ \tau(B_{j'}) &= 4\sin^2\left(\frac{\pi n'_{j'} \gamma'_{j'}}{\alpha'_{j'}}\right) D^{-1}_{H_*(B_{j'})}, \end{aligned} \tag{3.5}$$

where $\beta_j \gamma_j = 1 \bmod \alpha_j$, $\beta'_{j'} \gamma'_{j'} = 1 \bmod \alpha'_{j'}$, and $\gamma_j, \gamma'_{j'}$ are given by the twists of the exceptional fibres.

For X_i, $Y_{i'}$ which are homeomorphic to $P_i \times S^1$, $P'_{i'} \times S^1$, we can see that the S^1-component has the trivial holonomy, since the given representations take h, k into $\pm 1 \in SU(2)$. Hence we have that

$$\begin{aligned} \tau(X_i) &= (x_i^1 \otimes x_i^2 \otimes x_i^3) \otimes ((x_i^1 \cup h) \otimes (x_i^2 \cup h) \otimes (x_i^3 \cup h))^{-1} \\ &= D^{-1}_{H_*(X_i)}, \\ \tau(Y_{i'}) &= (y_{i'}^1 \otimes y_{i'}^2 \otimes y_{i'}^3) \otimes ((y_{i'}^1 \cup k) \otimes (y_{i'}^2 \cup k) \otimes (y_{i'}^3 \cup k))^{-1} \\ &= D^{-1}_{H_*(Y_{i'})}, \end{aligned} \tag{3.6}$$

where $\partial P_i = \{x_i^1, x_i^2, x_i^3\}$ and $\partial P'_{i'} = \{y_{i'}^1, y_{i'}^2, y_{i'}^3\}$. We omit the notation of invariant vectors under $\mathrm{Ad}(\rho(x_i^{\cdot}))$, $\mathrm{Ad}(\rho(y_{j'}^{\cdot}))$ for convenience. For the tori T_i, $T'_{i'}$, $T_{i_{\cdot}i_{\cdot\cdot}}$, $T'_{i'_{\cdot}i'_{\cdot\cdot}}$, T, we have the value of R-torsion for these tori from (2.2).

Now we compute the spectral sequence terms $E^r_{\cdot}$. First $E^1_{\cdot}$ is given by

$$\begin{array}{ccccc}
0 \longleftarrow & \bigoplus_{i=1}^{l_1} H_2(Z_i) & \xleftarrow{\delta_2} & \bigoplus_{i=1}^{l_2} H_2(T_i^2) & \longleftarrow 0 \\
0 \longleftarrow & \bigoplus_{i=1}^{l_1} H_1(Z_i) \bigoplus_{i=1}^{m+n} H_1(C_i) & \xleftarrow{\delta_1} & \bigoplus_{i=1}^{l_2} H_1(T_i^2) & \longleftarrow 0 \\
0 \longleftarrow & \bigoplus_{i=1}^{m+n} H_0(C_i) & \xleftarrow{\delta_0} & \bigoplus_{i=1}^{l_2} H_0(T_i^2) & \longleftarrow 0,
\end{array}$$

where $l_1 = 2(g+g'-1)+(m+n)$, $l_2 = 3(g+g'-1)+2(m+n)$ and

$$\bigcup_i^{l_1} Z_i = \bigcup_i^{2g-1+m} X_i \bigcup_{i'}^{2g'-1+n} Y_{i'},$$

$$\bigcup_i^{m+n} C_i = \bigcup_j^m A_j \bigcup_{j'}^n B_{j'},$$

$$\bigcup_{i=1}^{l_2} T_i^2 = \bigcup_{j=1}^m T_j \bigcup_{j'=1}^n T'_{j'} \bigcup_{\{i_{\cdot}i_{\cdot\cdot}\}} T_{i_{\cdot}i_{\cdot\cdot}} \bigcup_{\{i'_{\cdot}i'_{\cdot\cdot}\}} T'_{i'_{\cdot}i'_{\cdot\cdot}} \cup T.$$

We can see that

$$\delta_0(e) = e \quad \text{for} \quad e \in \bigoplus_i H_0(T_i) \bigoplus_{i'} H_0(T'_{i'}),$$

(3.7)

$$\delta_0(e) = 0 \quad \text{for} \quad e \in \bigoplus_{\{i_{\cdot}i_{\cdot\cdot}\}} H_0(T_{i_{\cdot}i_{\cdot\cdot}}) \bigoplus_{\{i'_{\cdot}i'_{\cdot\cdot}\}} H_0(T'_{i'_{\cdot}i'_{\cdot\cdot}}) \oplus H_0(T).$$

Hence the dimension of the kernel of δ_0 is $3(g+g'-1)+(m+n)$.

Since every gluing is made from the torus automorphism, we have that

$$\delta_2(l \cup m) = l \cup m \quad \text{for a basis of} \quad l \cup m \in \bigoplus_i^{l_2} H_2(T_i^2).$$

Hence the dimension of cokernel of δ_2 is $3(g+g'-1)+(m+n)$.

The map δ_1 depends on the gluing-torus automorphism more explicitly. To describe this map, we introduce a natural basis of $\bigoplus_{i=1}^{l_2} H_1(T_i^2)$ such that

$$H_1(T_j) = R[l_j, m_j], \quad H_1(T'_{j'}) = R[l'_{j'}, m'_{j'}],$$

$$H_1(T_{i.i..}) = R[l_{i.i..}, m_{i.i..}], \quad H_1(T'_{i'.i'..}) = R[l'_{i'.i'..}, m'_{i'.i'..}],$$

$$H_1(T) = R[l, m],$$

where R is the real number field. Then we have that

$$\begin{aligned} &\delta_1(m_j) = \alpha_j x^{\cdot}_i \oplus 0 \in H_1(X_i) \oplus H_1(A_j), \\ (3.8) \qquad &\delta_1(l_j) = \gamma_j x^{\cdot}_i \oplus a_j \in H_1(X_i) \oplus H_1(A_j) \quad \text{for some } i, \end{aligned}$$

where i is given such that A_j meets X_i and γ_j is given by the gluing torus automorphism as (3.5). For other intersection tori, we have that

$$\begin{aligned} &\delta_1(m_{i.i..}) = x^{\cdot}_{i.} \oplus x^{\cdot}_{i..} \in H_1(X_{i.}) \oplus H_1(X_{i..}), \\ (3.9) \qquad &\delta_1(l_{i.i..}) = 0 \in H_1(X_{i.}) \oplus H_1(X_{i..}). \end{aligned}$$

Similar formulas hold in the gluing of the part of M_2. And finally we have that

$$\delta_1(m) = x^{\cdot}_1 \oplus \alpha y^{\cdot}_1 \in H_1(X_1) \oplus H_1(Y_1),$$

$$\delta_1(l) = 0 \oplus \gamma y^{\cdot}_1 \in H_1(X_1) \oplus H_1(Y_1).$$

The kernel of δ_1 is generated by

$$\begin{aligned} &\{l_{i.i..}, l'_{i'.i'..}\} \quad \text{if} \quad \gamma \neq 0 \\ (3.10) \qquad &\{l_{i.i..}, l'_{i'.i'..}, l\} \quad \text{if} \quad \gamma = 0. \end{aligned}$$

Hence the dimension of the kernel of δ_1, which is same as the dimension of the cokernel of δ_1, is $3(g + g' - 1) + (m + n)$ or $3(g + g') + (m + n) - 4$ if $\gamma = 0$ or $\gamma \neq 0$.

From above we can see how $E^2_{\cdot}$ is given. In fact $E^2_{\cdot}$ is the $H_*(N)$. If we gather (3.5) $\sim$ (3.10) and apply these to (3.4), then we get the result about R-torsion in the proposition. Moreover we can see that $h_1(N_\gamma)$ is the sum of the dimension of the kernel of δ_0 and the dimension of the cokernel of δ_1, and that $h_2(N_\gamma)$ is the sum of the dimension of the kernel of δ_1 and the dimension of the cokernel of δ_2 in $E^1_{\cdot}$. □

§4. Half-density volumes of representation spaces. In this section we compute the half-density volumes of the irreducible $SU(2)$-representation spaces of Seifert fibred manifold M and graph manifold N. The half-density volume comes from the R-torsion, more precisely, from the determinant term of the first homology in the R-torsion. The tangent space of $R(M, SU(2))^-$ at $[\rho_M]$ can be identified with the first cohomology $H^1(M, su(2)_{\rho_M})$. So the determinant of the first homology $D_{H_1(M,su(2)_{\rho_M})}$ gives a volume form of $R(M, SU(2))^-$.

As we studied in Section 3, the sets of the rotation numbers $\{n_j\}, \{n_j, n'_{j'}\}$ of the $SU(2)$-representations ρ_M, ρ_N determine a connected component of the irreducible $SU(2)$-representation spaces $R(M, SU(2))^-, R(N, SU(2))^-$. We denote the connected component of irreducible $SU(2)$-representation spaces determined by $\{n_j\}, \{n_j, n'_{j'}\}$ by $R(M, (n_j))$, $R(N, (n_j, n'_{j'}))$. Hence we have the following equalities about the volume of $R(M, SU(2))^-$, $\mathrm{Vol}(R(M, SU(2))^-)$ for the Seifert fibred manifold M:

$$\begin{aligned}\mathrm{Vol}(R(M, SU(2))^-) &= \int_{R(M,SU(2))^-} \tau(M, \mathrm{Ad}(\rho_M))^{1/2} \\ &= \sum_{\{n_j\}} \int_{R(M,(n_j))} \tau(M, \mathrm{Ad}(\rho_M))^{1/2} \\ &= 2^m \sum_{\{n_j\}} \prod_{j=1}^{m} \frac{\left|\sin\left(\frac{2\pi n_j \beta_j^*}{\alpha_j}\right)\right|}{|\alpha_j|^{1/2}} \int_{R(M,(n_j))} D_{H_1(M,su(2)_{\rho_M})}.\end{aligned}$$

The similar formula holds for the graph manifold N. Hence we restrict our concern to the connected component $R(M, (n_j))$, $R(N, (n_j, n'_{j'}))$.

Before we compute the term $\int_{R(M,(n_j))} D_{H_1(M,su(2)_{\rho_M})}$, we remark on some facts. The spaces over which we integrate are noncompact open manifolds. We will use only the totally irreducible representations in the process of the integration to get the half-density volume of the representation space. This is possible because the complement of the totally irreducible representations are lower-dimensional, so that the complement can be considered as the measure zero set. Hence we can get the volume if we use only the totally irreducible representations with respect to the given decomposition in Section 3.

We shall use the generic fibration structure of $R(M, (n_j))$, $R(N, (n_j, n'_{j'}))$ to integrate the half-density volume form. So we introduce the generic fibration structures of $R(M, (n_j))$, $R(N, (n_j, n_{j'}))$.

Recall the base surface Σ_g of M. We delete the m-disks D_j^2's in Σ_g where ∂D_j^2 is free homotopic to q_j. We denote this by $\Sigma_{g,m}$. From the condition $q_j^{\alpha_j} h^{\beta_j} = 1$ we can give the holonomy conditions on $\partial\Sigma_{g,m}$. We denote the space of the irreducible $SU(2)$-representations of $\pi_1(\Sigma_{g,m})$ satisfying above holonomy conditions by $R(\Sigma_{g,m}, (n_j))$, where (n_j) can be determined as the rotation numbers which determine the connected component of $R(M, SU(2))^-$. In fact, we see easily that $R(\Sigma_{g,m}, (n_j))$ can be identified with $R(M, (n_j))$, so we use this identification from now on.

We decompose $\Sigma_{g,m}$ into the $2(g-1)+2m$-pairs of pants P_i with $3(g-1)+m$ intersection circles S_i^1 between the pair of pants such that $X_i = P_i \times S^1$ for the fibre circle S^1. We denote the $SU(2)$-representation space over S_i^1 by $\mathscr{L}_i$ as in Section 2. Then there is a natural map

$$p\colon R(\Sigma_{g,m}, (n_j)) \to \mathscr{L}_1 \times \cdots \times \mathscr{L}_{3(g-1)+m}$$

induced by the restriction such that

$$p([\rho]) = ([\rho|_{S_1^1}], \ldots, [\rho|_{S^1_{3(g-1)+m}}]).$$

We consider the $SU(2)$-representation $\rho_{\Sigma_{g,m}}$ of $\pi_1(\Sigma_{g,m})$ such that the representation $\rho|_{P^i}$ induced by the restriction to each piece P_i is irreducible. Such a representation of $\pi_1(\Sigma_{g,m})$ corresponds to a totally irreducible representation of $\pi_1(M)$ with respect to the corresponding decomposition of M. For such a representation $\rho_{\Sigma_{g,m}}$, the inverse image of $p(\rho_{\Sigma_{g,m}})$ can be identified with the $3(g-1)+m$ copies of the maximal torus T^1 divided by $(2g+m-3)$ copies of the center Z_2 of $SU(2)$. The maximal torus T^1 gives the gluing data between the pair of pants, and the center Z_2 comes from some symmetry. For the details about the proof of this fact, see [JW2, Proposition 3.8] and [W2, (4.65)]. So we have a fibration structure of $R(\Sigma_{g,m}, (n_j)) = R(M, (n_j))$ generically such that

$$0 \to \bigoplus_{i=1}^{3(g-1)+m} T_i^1 \Big/ \bigoplus_{i=1}^{2g+m-3} Z_{2_i} \to R(M, (n_j)) \to \bigoplus_{i=1}^{3(g-1)+m} \mathscr{L}_i \to 0. \tag{4.1}$$

The "generic fibration" means that the fibration exists only for the totally irreducible representations.

In a similar way, we can consider a generic fibration for the graph manifold N_γ. But we need to distinguish the cases when $\gamma = 0$ and $\gamma \neq 0$ for graph manifolds. We recall (3.2)

$$\phi_1 = \alpha\phi_2 + \beta\psi_2, \qquad \psi_1 = \gamma\phi_2 + \delta\psi_2, \tag{3.2}$$

where ψ_i is fixed for each connected component $R(N, (n_j, n'_{j'}))$. If $\gamma = 0$, we can see that

$$\psi_1 = \pm\psi_2 \quad \text{and} \quad \phi_1 = \pm\phi_2 + \beta\psi_2,$$

since $\alpha\delta = -1$. So the variables ϕ_1, ϕ_2 are related by

$$\phi_1 = \pm\phi_2 + \beta\psi_2.$$

If $\gamma \neq 0$, we have that

$$\phi_2 = \frac{1}{\gamma}\psi_1 - \frac{\delta}{\gamma}\psi_2.$$

If we combine this with the first equality of (3.2), then we have that

$$\phi_1 = \frac{\alpha}{\gamma}\psi_1 + \frac{1}{\gamma}\psi_2,$$

so the variables ϕ_1, ϕ_2 are fixed.

We can consider the base surface $\Sigma_{g,1}$ of M_1 and the base surface $\Sigma_{g',1}$ of M_2. We delete m-disks, n-disks from $\Sigma_{g,1}, \Sigma_{g',1}$, as above. Then we get $\Sigma_{g,m+1}, \Sigma_{g',n+1}$. Note the boundaries of $\Sigma_{g,m+1}, \Sigma_{g',n+1}$ consist of $m+1$ circles, $n+1$ circles, respectively. We decompose $\Sigma_{g,m+1}, \Sigma_{g',n+1}$ into

$$(2g+m-1)\text{-pair of pants } P_i \quad \text{and} \quad (2g'+n-1)\text{-pair of pants } P'_{i'}$$

such that $X_i = P_i \times S^1$, $Y_{i'} = P'_{i'} \times S^1$. Recall that $X_1 = P_1 \times S^1$ meets M_2 and $Y_1 = P'_1 \times S^1$ meets M_1.

As above, we may have the generic fibration of $R(N_\gamma, SU(2))^-$ from the correspondence of $\Sigma_{g,m+1}$ and $\Sigma_{g',n+1}$ to M_1 and M_2. But the generic fibration of $R(N_0, (n_j, n'_{j'}))$ is not the same as the generic fibration of $R(N_\gamma, (n_j, n'_{j'}))$ for $\gamma \neq 0$. If we consider the boundaries of P_1 and P'_1 around which the holonomies of $\rho|_{P_1}$, $\rho|_{P'_1}$ are ϕ_1, ϕ_2, respectively, then we know that ϕ_i is fixed if $\gamma \neq 0$ and is free with the above relation if $\gamma = 0$. Hence we have generic fibrations of $R(N_0, (n_j, n'_{j'}))$, $R(N_{\gamma\neq 0}, (n_j, n'_{j'}))$ such that

$$(4.2)\qquad \begin{aligned} 0 \to \bigoplus_{i=1}^{l_3} T_i^1 \Big/ \bigoplus_{i=1}^{l_4} Z_{2_i} \to R(N_0, SU(2), (n_j, n'_{j'})) \to \bigoplus_{i=1}^{l_5(0)} \mathcal{L}_i \to 0 \\ 0 \to \bigoplus_{i=1}^{l_3} T_i^1 \Big/ \bigoplus_{i=1}^{l_4} Z_{2_i} \to R(N_{\gamma\neq 0}, SU(2), (n_j, n'_{j'})) \to \bigoplus_{i=1}^{l_5(1)} \mathcal{L}_i \to 0, \end{aligned}$$

where $l_3 = 3(g+g'-1)+(m+n)$, $l_4 = 2(g+g')+(m+n)-3$, and $l_5(0) = 3(g+g'-1)+(m+n)$, $l_5(1) = 3(g+g')+(m+n)-4$.

To state our main result, we define a function $\zeta_A^k(s)$ defined for $s \in C$ with $\mathrm{Re}(s) \gg 0$, a finite set $A \in R - Z$, and for $k = 0, 1$. The function $\zeta_A^k(s)$ is defined by

$$\zeta_A^k(s) = \sum_{n=1}^{\infty} \left(\frac{(-1)^{(n+1)^k}}{n^s} \prod_{i=1}^{m} \left| \frac{\sin(\pi n a_i)}{\sin(\pi a_i)} \right| \right)$$

for $A = \{a_1, \ldots, a_m\}$. For an empty set ϕ, $\zeta_\phi^0(s)$ is the Riemann-zeta function.

THEOREM 4.3. *The half-density volume of $R(M,(n_j))$ is given by*

$$\mathrm{Vol}(R(M,(n_j))) = c_M \prod_{j=1}^{m} \frac{\left|\sin\left(\frac{\pi n_j \beta_j^*}{\alpha_j}\right)\right|}{|\alpha_j|^{1/2}} \zeta_{A_M}^0(2(g-1)+m),$$

where

$$c_M = 2^{m-1}\,\mathrm{Vol}(S^2)^m\,\mathrm{Vol}(SU(2))^{-(g-1+m)},$$

$$A_M = \left\{\frac{n_1}{\alpha_1},\ldots,\frac{n_m}{\alpha_m}\right\}.$$

The half-density volume of $R(N_\gamma,(n_j,n'_{j'}))$ is given by

$$\mathrm{Vol}(R(N_0,(n_j,n'_{j'}))) = c_N^0 \prod_{j=1}^{m} \frac{\left|\sin\left(\frac{\pi n_j \beta_j^*}{\alpha_j}\right)\right|}{|\alpha_j|^{1/2}} \prod_{j'=1}^{n} \frac{\left|\sin\left(\frac{\pi n'_j \beta_{j'}^*}{\alpha'_{j'}}\right)\right|}{|\alpha'_{j'}|^{1/2}} \zeta_{A_N}^k(2(g+g'-1)+(m+n)),$$

where

$$c_N^0 = 2^{m+n+1}\mathrm{Vol}(S^2)^{m+n}\mathrm{Vol}(SU(2))^{-(g+g'+m+n-1)},$$

$$k = \begin{cases} 0 & \text{if} \quad \beta\psi_2 \in Z \\ 1 & \text{if} \quad \beta\psi_2 \in Z[\frac{1}{2}] - Z, \end{cases}$$

$$A_N = \left\{2\phi_1, 2\phi_2, \frac{n_1}{\alpha_1},\ldots,\frac{n_m}{\alpha_m},\frac{n'_1}{\alpha'_1},\ldots,\frac{n'_n}{\alpha'_n}\right\},$$

and

$$\mathrm{Vol}(R(N_{\gamma\neq 0},(n_j,n'_{j'}))) = c_N^1 \frac{1}{|\gamma|^{1/2}} \prod_{j=1}^{m} \frac{\left|\sin\left(\frac{\pi n_j \beta_j^*}{\alpha_j}\right)\right|}{|\alpha_j|^{1/2}} \prod_{j'=1}^{n} \frac{\left|\sin\left(\frac{\pi n'_j \beta_{j'}^*}{\alpha'_{j'}}\right)\right|}{|\alpha'_{j'}|^{1/2}} \zeta_{A_{M_1}}^0(2g+m-1)\zeta_{A_{M_2}}^0(2g'+n-1),$$

where

$$c_N^1 = 2^{m+n+1}\,\mathrm{Vol}(S^2)^{m+n+2}\,\mathrm{Vol}(SU(2))^{-(g+g'+m+n)},$$

$$A_{M_1} = \left\{2\phi_1, \frac{n_1}{\alpha_1},\ldots,\frac{n_m}{\alpha_m}\right\}, \quad A_{M_2} = \left\{2\phi_2, \frac{n'_1}{\alpha'_1},\ldots,\frac{n'_n}{\alpha'_n}\right\}.$$

Proof. We shall compute only the half-density volume of $R(N_\gamma, (n_j, n'_{j'}))$. For $R(M, (n_j))$, we can get the result by the same way.

From Proposition 3.3 and (4.2), we have that

$$\begin{aligned}\operatorname{Vol}(R(N_\gamma, (n_j, n'_{j'}))) &= \int_{R(N_\gamma,(n_j,n'_{j'}))} \tau(N_\gamma, \operatorname{Ad}(\rho_{N_\gamma}))^{1/2} \\ &= 2^{m+n} f(\gamma)^{1/2} \prod_{j=1}^{m} \frac{\left|\sin\left(\frac{\pi n_j \beta_j^*}{\alpha_j}\right)\right|}{|\alpha_j|^{1/2}} \prod_{j'=1}^{n} \frac{\left|\sin\left(\frac{\pi n'_{j'} \beta_j'^*}{\alpha'_{j'}}\right)\right|}{|\alpha'_{j'}|^{1/2}} \int_{R(N_\gamma,(n_j,n'_{j'}))} D_{H_1(N_\gamma, su(2)_{\rho_N})} \\ &= 2^{m+n} f(\gamma)^{1/2} \prod_{j=1}^{m} \frac{\left|\sin\left(\frac{\pi n_j \beta_j^*}{\alpha_j}\right)\right|}{|\alpha_j|^{1/2}} \prod_{j'=1}^{n} \frac{\left|\sin\left(\frac{\pi n'_{j'} \beta_j'^*}{\alpha'_{j'}}\right)\right|}{|\alpha'_{j'}|^{1/2}} \int_F \mu_F \cdot \int_{B_\gamma} \mu_{B_\gamma},\end{aligned}$$

where $F = \bigoplus_{i=1}^{l_3} T_i^1 / \bigoplus_{i=1}^{l_4} Z_{2_i}$, $B_\gamma = \bigoplus_{i=1}^{l_5(k)} \mathscr{L}_i$ with $k = 0, 1$ if $\gamma = 0, \gamma \neq 0$, and μ_F, μ_{B_γ} are the volume forms of F, B_γ such that $D_{H_1(N_\gamma, su(2)_{\rho_N})} = \mu_F \cdot p^* \mu_{B_\gamma}$. We see easily that μ_F on F is the $3(g + g' - 1) + (m + n)$ copies of the natural volume form v of the maximal torus T^1 with $\int_{T^1} v = 1$. In fact μ_F is given by the volume form of the kernel of δ_0 in the proof of Proposition 3.3. So we have

$$\int_F \mu_F = \frac{1}{2^{2(g+g')+(m+n)-3}}.$$

The volume form μ_{B_γ} on $B_\gamma = \bigoplus_{i=1}^{l_5(k)} \mathscr{L}_i$ is given by the volume form of the cokernel of δ_1 in the proof of Proposition 3.3. So μ_{B_γ} can be written by $f_0 v_1 \cdots v_{l_5(0)}$ or $f_1 v_0 \cdots v_{l_5(1)}$ for some $f_k \in L^2(\bigoplus_{i=1}^{l_5(k)} \mathscr{L}_i)$ for $k = 0, 1$. From (2.4), Proposition 2.5, and the pants decomposition of $\Sigma_{g,m+1}, \Sigma_{g',n+1}$, we can see that $f_0 = f \cdot g$, $f_1 = f \cdot h$, where f is the product of

$$2(g + g' - 2) - \text{copies of } \frac{2}{\operatorname{Vol}(SU(2))^2} \sum_\alpha \frac{1}{n_\alpha} \prod_{i=1}^{3} \chi_\alpha(\cdot),$$

$$m - \text{copies of } \frac{2 \operatorname{Vol}(S^2)}{\operatorname{Vol}(SU(2))^2} \sum_\alpha \frac{1}{n_\alpha} \frac{\left|\sin\left(\frac{\pi n_\alpha n_j}{\alpha_j}\right)\right|}{\left|\sin\left(\frac{\pi n_j}{\alpha_j}\right)\right|} \chi_\alpha(\cdot)\chi_\alpha(\cdot),$$

$$n - \text{copies of } \frac{2 \operatorname{Vol}(S^2)}{\operatorname{Vol}(SU(2))^2} \sum_\alpha \frac{1}{n_\alpha} \frac{\left|\sin\left(\frac{\pi n_\alpha n'_{j'}}{\alpha'_{j'}}\right)\right|}{\left|\sin\left(\frac{\pi n'_{j'}}{\alpha'_{j'}}\right)\right|} \chi_\alpha(\cdot)\chi_\alpha(\cdot),$$

and g is the product of

$$2-\text{copies of } \frac{2}{\mathrm{Vol}(SU(2))^2}\sum_{\alpha}\frac{1}{n_\alpha}\prod_{i=1}^{3}\chi_\alpha(\cdot),$$

and h is the product of

$$\frac{2\,\mathrm{Vol}(S^2)}{\mathrm{Vol}(SU(2))^2}\sum_{\alpha}\frac{1}{n_\alpha}\frac{|\sin(2\pi n_\alpha\phi_1)|}{|\sin(2\pi\phi_1)|}\chi_\alpha(\cdot)\chi_\alpha(\cdot),$$

$$\frac{2\,\mathrm{Vol}(S^2)}{\mathrm{Vol}(SU(2))^2}\sum_{\alpha}\frac{1}{n_\alpha}\frac{|\sin(2\pi n_\alpha\phi_2)|}{|\sin(2\pi\phi_2)|}\chi_\alpha(\cdot)\chi_\alpha(\cdot),$$

where the value of representation $\rho_{N_\gamma}([x_i^{\cdot}])$ or $\rho_{N_\gamma}([y_{i'}^{\cdot}])$ appears at the slot of $\chi_\alpha(\cdot)$ if $x_i^{\cdot}$ or $y_{i'}^{\cdot}$ is a homotopy class of a boundary of P_i or $P_{i'}'$.

We use the orthogonal pairing of the characters on each intersection circle between a pair of pants such that

$$\int_{\mathscr{L}_i}\chi_\alpha(\rho_N(\cdot))\chi_\beta(\rho_N(\cdot)^{-1})v_i=\delta_{\alpha,\beta}\,\mathrm{Vol}(SU(2)).$$

These pairings on $\mathscr{L}_i$ for $1\leqslant i\leqslant l_5(1)$ with respect to v_i give $\int_{B_\gamma}\mu_{B_\gamma}$ for $\gamma\neq 0$, so that we get the volume of $R(N_\gamma,(n_j,n_{j'}'))$ when $\gamma\neq 0$. Note that this pairing does not occur on the intersecting circle between P_1 and P_1' since the holonomies ϕ_1 and ϕ_2 are fixed if $\gamma\neq 0$.

When $\gamma=0$, the above pairings on $\mathscr{L}_i$ with respect to v_i give the same result, except for one $\mathscr{L}_i$, which comes from the intersection of P_1 and P_1'. This is because the holonomies of ρ_{N_γ} around a boundary of P_1 and P_1' which give the pairing do not coincide but have the relation $\phi_1=\pm\phi_2+\beta\psi_2$. So the pairing in this case is given by

$$\int_{\mathscr{L}_i}\chi_\alpha(\rho_N(D(e^{2\pi\phi_1})))\chi_\beta(\rho_N(D(e^{2\pi\phi_2}))^{-1})v_i=(-1)^k\delta_{\alpha,\beta}\,\mathrm{Vol}(SU(2)),$$

where $D(e^{2\pi\phi_i})$ is the $SU(2)$-matrix with the diagonal elements $\{e^{2\pi\phi_i},e^{-2\pi\phi_i}\}$ and

$$k=\begin{cases}0 & \text{if}\quad \beta\psi_2\in Z\\ 1 & \text{if}\quad \beta\psi_2\in Z[\tfrac{1}{2}]-Z.\end{cases}$$

Recall that β is given in (3.1). We have that $\{n_\alpha\}$ is the set of the natural numbers for the Lie group $SU(2)$. For this, see the Proposition 5.3 of [BtD]. The constants C_N^0, C_N^1 are given by gathering all the constants in the above pairings. □

We remark on some facts about Theorem 4.3. We compute the half-density volumes of $R(M,(n_j))$, $R(N_\gamma,(n_j,n'_{j'}))$ using the fibration structure of the representation space $R(\Sigma_{g,m},(n_j))$, which has the symplectic structure. In [W2] Witten computes the symplectic volume of $R(\Sigma_{g,m},(n_j))$. We use his method with some modification. We point out two differences between the symplectic volume form of $R(\Sigma_{g,m},(n_j))$ and the half-density of $R(M,(n_j))$.

The first comes from two different volume forms of the fibre $\bigoplus_{i=1}^{3(g-1)+m} T_i^1/\bigoplus_{i=1}^{2g+m-3} Z_{2_i}$. In our case the volume form over the fibre comes from the natural volume form v on the maximal torus T^1 of $SU(2)$, but in [W2] the fibre volume form comes from the volume form v_0 induced from the volume form of $SU(2)$. Of course, the difference comes from the different constructions of two volume forms.

The second is the term $\sin(\pi(n_j/\alpha_j))$. In [W2] this term is necessary to define natural symplectic volume form of $R(\Sigma_{g,m},(n_j))$ so that this term cancels out a factor of character term $\sin(\pi n n_j/\alpha_j)/\sin(\pi n_j/\alpha_j)$ in [W2]. But we do not need this term to give the cancelation.

§5. Application to the Jeffrey-Weitsman-Witten invariant. In this section we apply our result of the previous section to compute the Jeffrey-Weitsman-Witten invariant of a Seifert fibred manifold M with base surface $\Sigma_{g\geqq 2}$. In this case the irreducible $SU(2)$-representation space $R(M,SU(2))^-$ is a nondiscrete set, so that the R-torsion is used as the half-density of $R(M,SU(2))^-$ in defining the Jeffrey-Weitsman-Witten invariant.

We review the Chern-Simons gauge theory to understand the definition of the Jeffrey-Weitsman-Witten invariant. For details of the Chern-Simons gauge theory, see [RSW], [JW1].

Let X be a 2-dimensional manifold and P be a principal $SU(2)$-bundle over X. Let $\mathscr{A},\mathscr{A}_F,\mathscr{G}$ be the affine space of connection 1-forms of P, the space of flat connections of P, and the gauge transformation group of P, respectively. Let $\mathscr{M}_2$ be the moduli space of the flat connections of P.

We consider a 3-dimensional manifold Y_1 with a boundary X. Moreover, we assume that a neighborhood of X in Y_1 is diffeomorphic to $X\times[0,1)$. For $A\in\mathscr{A}, g\in\mathscr{G}$, we consider a $U(1)$-valued function $\mathscr{S}(A,g)$ defined by

$$\mathscr{S}(A,g)\equiv\exp(2\pi i(CS(\tilde{A}^{\tilde{g}})-CS(\tilde{A}))),$$

where $\tilde{A}$ and $\tilde{g}$ are the extensions of A and g into Y_1, $\tilde{A}^{\tilde{g}}$ is the gauge transformation of $\tilde{A}$ by $\tilde{g}$, and the Chern-Simons invariant $CS(\tilde{A})$ is given by

$$CS(\tilde{A})=\frac{1}{8\pi^2}\int_{Y_1} tr\left(d\tilde{A}\wedge\tilde{A}+\frac{2}{3}\tilde{A}\wedge\tilde{A}\wedge\tilde{A}\right).$$

Such an extension of $\tilde{g}$ always exists since $\pi_1(SU(2))=\pi_2(SU(2))=0$. We choose the extensions so that $\tilde{A}$ and $\tilde{g}$ are pullbacks of A and g by the projection to X over $X \times [0,1)$, respectively. Then $\mathscr{S}$ is independent on the extension Y_1 and the extension $\tilde{A}$ and $\tilde{g}$. In fact, the extensions $(\tilde{A}_1, \tilde{g}_1)$ and $(\tilde{A}_2, \tilde{g}_2)$ into Y_1 and Y_2 give a connection $\tilde{B}$ and a gauge transformation $\tilde{h}$ on $Y = Y_1 \cup Y_2$ so that

$$\exp(2\pi i(CS(\tilde{A}_1^{\tilde{g}}) - CS(\tilde{A}_1))) \exp(2\pi i(CS(\tilde{A}_2^{\tilde{g}}) - CS(\tilde{A}_2)))^{-1}$$
$$= \exp(2\pi i(CS(\tilde{B}^{\tilde{h}}) - CS(\tilde{B}))) = 1.$$

The above function $\mathscr{S}$ over $\mathscr{A} \times \mathscr{G}$ is a cocycle since

$$\mathscr{S}(\tilde{A}, \tilde{g})\mathscr{S}(\tilde{A}^{\tilde{g}}, \tilde{h}) = \mathscr{S}(\tilde{A}, \tilde{g}\tilde{h}).$$

We can define a line bundle $\mathscr{L}$ over $\mathscr{M}_2$ by

$$\mathscr{L} \equiv \mathscr{A}_F \times_{\mathscr{S}} C,$$

where the right side is the quotient space given by the equivalence relation

$$(A, z) \sim (A^g, \mathscr{S}(A, g)z)$$

for $A \in \mathscr{A}_F, z \in C$.

We consider a 3-dimensional manifold Y and a principal $SU(2)$-bundle P_Y over Y. We decompose Y into two handle bodies Y_1 and Y_2. Let X be the intersection of Y_1 and Y_2. We apply the above construction to 2-dimensional manifold X and $P_Y|_X = P$. We consider the restriction of line bundle $\mathscr{L}$ to Lagrangian submanifolds L_1, L_2 of $\mathscr{M}_2$, where L_1, L_2 are made from the handle bodies Y_1, Y_2 of Y. Let $\mathscr{L}_i$ be the restriction of the line bundle $\mathscr{L}$ to L_i. Then there is a section $\mathscr{S}_i(A)$ of $\mathscr{L}_i$ over L_i defined by

$$\mathscr{S}_i(A) = \exp(2\pi i CS(A_{Y_i}))$$

for $[A] \in \mathscr{M}_2$, and A_{Y_i} is an extension of A to Y_i.

Now we consider the intersection of the two Lagrangian submanifolds L_1, L_2 in $\mathscr{M}_2$. Then we can see that this intersection is the moduli space of flat connections of P_Y over Y, which we denote by $\mathscr{M}_3$. By the correspondence between the flat connection A and the $SU(2)$-representation ρ_Y of $\pi_1(Y)$, $\mathscr{M}_3$ can be identified with $R(Y, SU(2))$. There may occur singularities of $\mathscr{M}_2$ within $\mathscr{M}_3$. But the set of singularities is a measure zero set in $\mathscr{M}_3$. So we may not consider these singularities in the following construction, since we shall integrate over the dense subset $R(Y, SU(2))^-$ of $\mathscr{M}_3 = R(Y, SU(2))$.

We consider the k-tensor power of $\mathscr{L}$, $\mathscr{L}^{\otimes^k}$ over $\mathscr{M}_2$ and their restrictions to two Lagrangian submanifolds L_1, L_2. We denote these by $\mathscr{L}_1^{\otimes^k}, \mathscr{L}_2^{\otimes^k}$. Then we can pair two sections $\mathscr{S}_i^k = \mathscr{S}_i^{\otimes^k}$ of $\mathscr{L}_i^{\otimes^k}$ in $\mathscr{M}_3$ by the hermitian product of the complex line C. We denote this pairing by $\langle \mathcal{S}_1^k, \mathcal{S}_2^k \rangle$. Then this can be considered as an $U(1)$-valued function on $\mathcal{M}_3 = R(Y, SU(2))$. We can see easily that $\langle \mathcal{S}_1^k, \mathcal{S}_2^k \rangle$ at a connection A is the exponential of the Chern-Simons invariant of Y, that is, $\exp(2k\pi i CS(A))$.

We recall that the half-density derived from the R-torsion $\tau(Y, \mathrm{Ad}(\rho_Y))^{1/2}$ can be considered as a measure of $R(Y, SU(2))^-$. The Jeffrey-Weitsman-Witten invariant is defined by integrating the pairing $\langle \mathcal{S}_1^k, \mathcal{S}_2^k \rangle$ with respect to half-density $\tau(Y, \mathrm{Ad}(\rho_Y))^{1/2}$ over $R(Y, SU(2))^-$ using the correspondence between the flat connection A of P and the $SU(2)$-representation ρ_Y of $\pi_1(Y)$. We formulate this construction as the following definition.

Definition 5.1. For an integer k, the Jeffrey-Weitsman-Witten invariant $Z(Y, k)$ is defined by

$$Z(Y, k) = \int_{R(Y,SU(2))^-} \langle \mathcal{S}_1^k, \mathcal{S}_2^k \rangle \tau(Y, \mathrm{Ad}(\rho_Y))^{1/2}.$$

This definition is given in [JW1]. This definition is motivated from the asymptotic expansion of the Witten invariant $Z_Y(k)$ of 3-dimensional manifold Y [W1]. The asymptotic expansion of $Z_Y(k)$ is given by

$$Z_Y(k) \simeq \frac{1}{2} \sum_i (\tau(Y, A_i))^{1/2} \exp\left(- \frac{3\pi i + 2\pi i SF(A_i)}{4} \right) \exp(2(k+2)\pi i CS_Y(A_i)),$$

where the sum is taken over the finite set of flat connections A_i, $\tau(Y, A_i)$ is the Reidemeister torsion for A_i of Y, and $SF(A_i)$ is the spectral flow from trivial connection to the flat connection A_i. The above formula is given in [FG]. We can see that if the moduli space of flat connections is a discrete set, then Definition 5.1 is almost the same as the leading term of the above asymptotic expansion, since the the square root of the R-torsion becomes a point mass in this case.

Now we compute the Jeffrey-Weitsman-Witten invariant $Z(M, k)$ of Seifert fibred manifold M with the nondiscrete irreducible $SU(2)$-representation space $R(M, SU(2))^-$ by applying the previous result. To compute $Z(Y, k)$, we must integrate the pairing $\langle \mathcal{S}_1^k, \mathcal{S}_2^k \rangle$ with respect to the half-density of the R-torsion over $R(M, SU(2))^-$. We know that the value $\langle \mathcal{S}_1^k, \mathcal{S}_2^k \rangle$ at the flat connection A is simply the Chern-Simons invariant of A. This invariant is constant in each connected component $R(M, (n_j))$. By the result of [A], the value for the fixed connected component $R(M, (n_j))$ is given by

$$\exp\left[2k\pi i \left(- \sum_{j=1}^{m} \left(\frac{\beta_j^* n_j^2}{\alpha_j} + \frac{2\varepsilon n_j}{\alpha_j} \right) + \varepsilon^2 \sum_{j=1}^{m} \frac{\beta_j}{\alpha_j} \right) \right], \tag{5.2}$$

where $\varepsilon = 1/2, 1$ if $\rho_M(h) = -1, 1$, and $\beta_j \beta_j^* = 1 \pmod{\alpha_j}$ as above.

So the value of the Jeffrey-Weitsman-Witten invariant $Z(Y,k)$ over the connected component $R(M,(n_i))$ is given by

$$\int_{R(M,(n_j))} \langle \mathscr{S}_1^k, \mathscr{S}_2^k \rangle (\tau(M, \mathrm{Ad}(\rho_M)))^{1/2}$$

$$= 2^m \langle \mathscr{S}_1^k, \mathscr{S}_2^k \rangle \prod_{j=1}^m \frac{\left|\sin\left(\frac{\pi n_j \beta_j^*}{\alpha_j}\right)\right|}{|\alpha_j|^{1/2}} \int_{R(M,(n_j))} D_{H_1(M, su(2)_{\rho_M})}.$$

So we have the following theorem from Theorem 4.3 and (5.2).

THEOREM 5.3. *For the integer k, the Jeffrey-Weitsman-Witten invariant $Z(M,k)$ of the Seifert fibred manifold $M(g,(\alpha_1,\beta_1,\dots,\alpha_m,\beta_m))$ is given by*

$$c_M \sum_{\{(n_j)\}} \exp\left[2k\pi i\left(-\sum_{j=1}^m \left(\frac{\beta_j^* n_j^2}{\alpha_j} + \frac{2\varepsilon n_j}{\alpha_j}\right) + \varepsilon^2 \sum_{j=1}^m \frac{\beta_j}{\alpha_j}\right)\right]$$

$$\times \prod_{j=1}^m \frac{\left|\sin\left(\frac{\pi n_j \beta_j^*}{\alpha_j}\right)\right|}{|\alpha_j|^{1/2}} \sum_{n=1}^{\infty} \left(\frac{1}{n^{2(g-1)+m}} \prod_{j=1}^m \frac{\left|\sin\left(\frac{\pi n n_j}{\alpha_j}\right)\right|}{\left|\sin\left(\frac{\pi n_j}{\alpha_j}\right)\right|}\right),$$

where the above sum is taken over the finite set of the rotation numbers $\{(n_j)\}$, $c_M = 2^{m-1}\,\mathrm{Vol}(S^2)^m\,\mathrm{Vol}(SU(2))^{g+m-1}$, and $\varepsilon = 1/2$ or 1 if $\rho_M(h) = -1$ or 1.

We can see that $Z(M,k)$ depends only on the manifold M via the Seifert invariant $(g;(\alpha_i,\beta_i))$ since the set of all the rotation numbers (n_j) is determined by $\pi_1(M)$.

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