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ETA INVARIANTS AND REGULARIZED DETERMINANTS FOR ODD DIMENSIONAL HYPERBOLIC MANIFOLDS WITH CUSPS

By Jinsung Park

Abstract. We study eta invariants of Dirac operators and regularized determinants of Dirac Laplacians over hyperbolic manifolds with cusps and their relations with Selberg zeta functions. Using the Selberg trace formula and a detailed analysis of the unipotent orbital integral, we show that the eta and zeta functions defined by the relative traces are regular at the origin so that we can define the eta invariant and the regularized determinant. We also show that the Selberg zeta function of odd type has a meromorphic extension over $\mathbb{C}$, prove a relation of the eta invariant and a certain value of the Selberg zeta function of odd type, and derive a corresponding functional equation. These results generalize the earlier work of John Millson to hyperbolic manifolds with cusps. We also prove that the Selberg zeta function of even type has a meromorphic extension over $\mathbb{C}$, relate it to the regularized determinant, and obtain a corresponding functional equation.

1. Introduction. In the seminal paper [12], Millson derived a relation of the eta invariant of the odd signature operator and a certain value of Selberg zeta function of odd type for compact hyperbolic manifolds of dimension $(4n-1)$. To prove this, Millson used the Selberg trace formula, which relates the spectral data to the geometric data, applied to a test function defined by the odd heat kernel of the odd signature operator. In [7], the corresponding work was done for the analytic torsion and the Ruelle zeta function for odd dimensional hyperbolic manifolds. These results have been generalized to the case of compact locally symmetric spaces of higher rank in [13], [14].

It would be an interesting problem to extend the aforementioned results to noncompact locally symmetric spaces with finite volumes. However, we encounter several serious difficulties when we discuss the extension of those results to noncompact locally symmetric spaces. First of all, the heat operator of the Laplacian is not of trace class. Hence we cannot use the trace of the heat operator as in the compact case. In [18], [19], Müller introduced the relative trace to overcome this kind of difficulty and defined corresponding relative eta invariants and relative regularized determinants. In this paper, we follow Müller's approach to study the (relative) spectral invariants for noncompact hyperbolic manifolds with finite volumes. If the continuous spectrum of the operator has a gap near zero, the relative trace behaves as the usual trace in the compact case. This is the

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case of the Laplacians acting on functions. The regularized determinants of these operators and their relations with Selberg zeta functions have been extensively studied for two and three-dimensional manifolds with cusps in [4], [5], [10], [11], [17]. However, if the continuous spectrum of the operator has no gap near zero, we need to know the large time behavior of the relative trace, so its relation with the scattering theory. In this paper, we study the spectral invariants of Dirac operators and Dirac Laplacians for odd dimensional hyperbolic manifolds with cusps, whose continuous spectrums reach zero.

Our approach is to use the Selberg trace formula applied to test functions defined by the heat kernels. To do so, we analyze the corresponding geometric side of the Selberg trace formula, in particular, the unipotent orbital integral. A detailed analysis of these terms enables us to show that the Selberg zeta functions of odd/even type have meromorphic extensions over $\mathbb{C}$. These results can be considered as generalizations of an old result of Gangolli and Warner in [8] to the case of nontrivial locally homogeneous vector bundles over noncompact locally symmetric spaces.

We explain our result more precisely. Let $X = \Gamma\backslash \mathrm{Spin}(2n+1,1)/\mathrm{Spin}(2n+1)$ be a $(2n+1)$-dimensional hyperbolic manifold with cusps. Here Γ is a discrete subgroup of $G = \mathrm{Spin}(2n+1,1)$ with finite co-volume. Throughout this paper, we also assume that the group generated by the eigenvalues of Γ contains no root of unity. We now consider the Dirac operator $\mathcal{D}$ acting on $L^2(X,E)$. Here the spinor bundle E over X is a locally homogeneous vector bundle defined by the spin representation τ_n of the maximal compact subgroup $\mathrm{Spin}(2n+1)$ of $\mathrm{Spin}(2n+1,1)$. Let us observe that the restriction of τ_n to $\mathrm{Spin}(2n) \subset \mathrm{Spin}(2n+1)$ has the decomposition $\sigma_+ \oplus \sigma_-$ where $\sigma_\pm$ denotes the half spin representation of $\mathrm{Spin}(2n)$. Let $\mathcal{K}_t$ be the family of functions over $G = \mathrm{Spin}(2n+1,1)$ given by taking the local trace of the integral kernel of $e^{-t\widetilde{D}^2}$ or $\widetilde{D}e^{-t\widetilde{D}^2}$ where $\widetilde{D}$ is the lifting of $\mathcal{D}$ over the universal covering space of X. Now the Selberg trace formula applied to $\mathcal{K}_t$ has the following form,

$$\begin{aligned}\sum_{\sigma=\sigma_\pm}\sum_{\lambda_k\in\sigma_p^\pm}\widehat{\mathcal{K}}_t(\sigma,i\lambda_k) - \frac{1}{4\pi}\int_{-\infty}^{\infty}\mathrm{Tr}(C_\Gamma(\sigma_+,-i\lambda)C'_\Gamma(\sigma_+,i\lambda)\pi_\Gamma(\sigma_+,i\lambda)(\mathcal{K}_t))\,d\lambda \\ = I_\Gamma(\mathcal{K}_t)+H_\Gamma(\mathcal{K}_t)+U_\Gamma(\mathcal{K}_t)\end{aligned}$$

where $\sigma_p := \sigma_p^+ \cup \sigma_p^-$ gives the point spectrum of $\mathcal{D}$, $C_\Gamma(\sigma_+, i\lambda)$ is the intertwining operator (note that $C_\Gamma(\sigma_+,i\lambda) = C_\Gamma(\sigma_-,i\lambda)$ since σ_+,σ_- are unramified) and $I_\Gamma(\cdot)$, $H_\Gamma(\cdot)$, $U_\Gamma(\cdot)$ are the identity, hyperbolic and unipotent orbital integrals, respectively. Following [15], [18], [19], we define certain operators $\mathcal{D}_0(i)$ determined by $\mathcal{D}$ such that $e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa} e^{-t\mathcal{D}_0(i)^2}$, $\mathcal{D}e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa}\mathcal{D}_0(i)e^{-t\mathcal{D}_0^2(i)}$ are trace class operators on $L^2(X,E)$ where κ is the number of the cusps of X. We also show that the spectral side of the above Selberg trace formula is equal to the relative trace $\mathrm{Tr}(e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa} e^{-t\mathcal{D}_0^2(i)})$ or $\mathrm{Tr}(\mathcal{D}e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa}\mathcal{D}_0(i)e^{-t\mathcal{D}_0^2(i)})$; for example, if $\mathcal{K}_t$ is give by $e^{-t\widetilde{\mathcal{D}}^2}$,

$$\begin{aligned}\operatorname{Tr}&\left(e^{-t\mathcal{D}^2}-\sum_{i=1}^{\kappa}e^{-t\mathcal{D}_0^2(i)}\right)\\&=\sum_{\sigma=\sigma\pm}\sum_{\lambda_k\in\sigma_p^{\pm}}\widehat{\mathcal{K}}_t(\sigma,i\lambda_k)-\frac{1}{4\pi}\int_{-\infty}^{\infty}\operatorname{Tr}(C_\Gamma(\sigma_+,-i\lambda)C'_\Gamma(\sigma_+,i\lambda)\pi_\Gamma(\sigma_+,i\lambda)(\mathcal{K}_t))d\lambda\\&=I_\Gamma(\mathcal{K}_t)+H_\Gamma(\mathcal{K}_t)+U_\Gamma(\mathcal{K}_t).\end{aligned}$$

A similar formula holds for $\operatorname{Tr}(\mathcal{D}e^{-t\mathcal{D}^2}-\sum_{i=1}^{\kappa}\mathcal{D}_0(i)e^{-t\mathcal{D}_0^2(i)})$ with the corresponding kernel function $\mathcal{K}_t$. The exact forms of $I_\Gamma(\mathcal{K}_t)$, $H_\Gamma(\mathcal{K}_t)$ are well-known, in particular, the hyperbolic orbital integrals $H_\Gamma(\mathcal{K}_t)$ provide us with the Selberg zeta function of odd/even type. Hence, a main task in this paper is the analysis of $U_\Gamma(\mathcal{K}_t)$. To do so, we use a result of Hoffmann in [9] and perform explicit computations for the weighted unipotent orbital integrals for our concerned cases. These explicit computations constitute some of the main ingredients of this paper. By these explicit computations, we can show that $U_\Gamma(\mathcal{K}_t)=0$ if $\mathcal{K}_t$ is determined by $\widetilde{D}e^{-t\widetilde{D}^2}$.

Following Müller's approach in [17], [18], [19], we define the eta function $\eta_\mathcal{D}(s)$ and the zeta function $\zeta_{\mathcal{D}^2}(s)$ using the relative traces $\operatorname{Tr}(e^{-t\mathcal{D}^2}-\sum_{i=1}^{\kappa}e^{-t\mathcal{D}_0^2(i)})$, $\operatorname{Tr}(\mathcal{D}e^{-t\mathcal{D}^2}-\sum_{i=1}^{\kappa}\mathcal{D}_0(i)e^{-t\mathcal{D}_0^2(i)})$, respectively (see (44), (30), (31) for the precise definitions of $\eta_\mathcal{D}(s)$, $\zeta_{\mathcal{D}^2}(s)$). The continuous spectrum of $\mathcal{D}$ is the whole real line, hence the large time contributions of the relative traces can also give rise to poles of $\eta_\mathcal{D}(s)$, $\zeta_{\mathcal{D}^2}(s)$. Therefore, we need to consider separately the small time contributions and the large time contributions for the meromorphic extensions of $\eta_\mathcal{D}(s)$, $\zeta_{\mathcal{D}^2}(s)$. We use the analytic expansion of the intertwining operator along the imaginary axis to get the large time contribution and we analyze all the terms in the geometric side of the Selberg trace formula for the small time contribution. We prove the following meromorphic structures of the eta and zeta functions.

THEOREM 1.1. *The eta function $\eta_\mathcal{D}(z)$ and the zeta function $\zeta_{\mathcal{D}^2}(z)$ have the meromorphic structures,*

$$\Gamma((z+1)/2)\,\eta_\mathcal{D}(z)=\sum_{k=0}^{\infty}\frac{-2\gamma_k}{z-2k-2}+K(z),$$

$$\Gamma(z)\,\zeta_{\mathcal{D}^2}(z)=\sum_{k=-n}^{\infty}\frac{\beta_k}{z+k-\frac{1}{2}}+\frac{\beta'_0}{(z-\frac{1}{2})^2}-\frac{h}{z}+\sum_{k=0}^{\infty}\frac{-\gamma'_k}{z-k-\frac{1}{2}}+H(z)$$

where β_k, β'_0 are locally computable constants, γ_k, γ'_k are constants which are determined by the intertwining operator $C_\Gamma(\sigma_+,i\lambda)$, h is the multiplicity of the zero *eigenvalues of $\mathcal{D}$, and $K(z)$, $H(z)$ are holomorphic functions. In particular, $\eta_{\mathcal{D}}(z)$ and $\zeta_{\mathcal{D}^2}(z)$ are regular at $z = 0$.*

It follows from Theorem 1.1 that we can define the eta invariant by

$$\eta(\mathcal{D}) := \eta_{\mathcal{D}}(0).$$

Following [12], we use the Selberg trace formula to derive the relation of the eta invariant $\eta(\mathcal{D})$ and the value of the Selberg zeta function of odd type $Z^o_H(s)$ at $s = n$ (see (45) for the precise definition of $Z^o_H(s)$). Let us remark that $Z^o_H(s)$ is defined a priori only for $\mathrm{Re}(s) \gg 0$ and the meromorphic extension of $Z^o_H(s)$ over $\mathbb{C}$ is one of the main results of this paper. As we mentioned above, we show that all unipotent terms are vanishing in the Selberg trace formula applied to the odd heat kernel function. In conclusion, the equality of the relative trace and the orbital integrals for the odd kernel function is exactly the same as in the case of compact hyperbolic manifolds. Therefore we can expect the same formula of $\eta(\mathcal{D})$ and $Z^o_H(s)$ as in compact hyperbolic manifolds. However, in the corresponding functional equation, the term determined by the intertwining operator appears. The following theorem states our results for $\eta(\mathcal{D})$ and $Z^o_H(s)$.

THEOREM 1.2. *The Selberg zeta function of odd type $Z^o_H(s)$ has a meromorphic extension over $\mathbb{C}$ with $s = n$ as a regular point and the following equalities hold:*

$$\eta(\mathcal{D}) = \frac{1}{\pi i} \log Z^o_H(n),$$

$$Z^o_H(n+s) Z^o_H(n-s) = \exp(2\pi i \eta(\mathcal{D})) \left(\frac{\det C_+(s) C_-(0)}{\det C_-(s) C_+(0)} \right)^{-2^{n-1}} \qquad \textit{for} \quad s \in \mathbb{C}.$$

Here $C_\pm(s)$ are linear operators given by $C_\Gamma(\sigma_+, s) = \begin{pmatrix} 0 & C_-(s) \\ C_+(s) & 0 \end{pmatrix}$.

We can define the zeta function $\zeta_{\mathcal{D}^2}(z, s)$ of the shifted Dirac Laplacian $\mathcal{D}^2 + s^2$ where s is a positive real number. Now the continuous spectrum of $\mathcal{D}^2 + s^2$ does not reach 0 and the large time contribution does not create any poles of $\zeta_{\mathcal{D}^2}(z, s)$. Therefore we can see that $\zeta_{\mathcal{D}^2}(z, s)$ is regular at $z = 0$ by Theorem 1.1. It follows that the regularized determinant

$$\mathrm{Det}(\mathcal{D}^2, s) := \exp(-\zeta'_{\mathcal{D}^2}(0, s))$$

is well defined. We show that $\mathrm{Det}(\mathcal{D}^2, s)$ can be extended to a meromorphic function of s on $\mathbb{C}$. Let us observe that $\mathrm{Det}(\mathcal{D}^2, s) \neq \mathrm{Det}(\mathcal{D}^2, -s)$ as a meromorphic function over $\mathbb{C}$ (see Remark 8.2). We use the Selberg trace formula to prove a relation between $\mathrm{Det}(\mathcal{D}^2, s)$ and the geometric data, which consists of the Selberg zeta function of even type $Z^e_H(s)$ (see (58) for the precise definition of $Z^e_H(s)$) and certain meromorphic functions over $\mathbb{C}$ derived from the identity and unipotent orbital integrals. Let us also remark that $Z^e_H(s)$ is defined a priori only for $\mathrm{Re}(s) \gg 0$ and its meromorphic extension over $\mathbb{C}$ is one of the main results of this paper. By a similar manner as in the previous case, we also derive a functional equation for $\mathrm{Det}(\mathcal{D}^2, s)$ and $Z^e_H(s)$ where the unipotent factor plays a nontrivial role. The following theorem states our results for $\mathrm{Det}(\mathcal{D}^2, s)$ and $Z^e_H(s)$.

THEOREM 1.3. *The Selberg zeta function of even type $Z^e_H(s)$ has a meromorphic extension over $\mathbb{C}$ and the following equalities hold for any $s \in \mathbb{C}$:*

$$\mathrm{Det}(\mathcal{D}^2, s) = C Z^e_H(s+n)\Gamma\left(s+\tfrac{1}{2}\right)^{-2^n\kappa} \exp\left(4\pi \int_0^s p(\sigma_+, i\lambda) + P_U(i\lambda)\, d\lambda\right),$$

$$\mathrm{Det}(\mathcal{D}^2, s)^2 = C^2(\det C_\Gamma(\sigma_+, s))^{-2^{n-1}} Z^e_H(n+s) Z^e_H(n-s) \left(\Gamma\left(s+\tfrac{1}{2}\right)\Gamma\left(-s+\tfrac{1}{2}\right)\right)^{-2^n\kappa}$$

where C is a constant, $p(\sigma_+, s)$ is Plancherel measure for σ_+, $P_U(s)$ is an even polynomial of degree $(2n-4)$, and κ denotes the number of the cusps of X.

If we compare Theorem 1.2 with Theorem 1.3, we can see that there are no defect terms determined by the cusps in the relation of $\eta(\mathcal{D})$ and $Z^o_H(n)$. But the defect terms appear in the relation of $\mathrm{Det}(\mathcal{D}^2, s)$ and $Z^e_H(s)$.

This paper has the following structure. In Section 2, we define a Dirac operator $\mathcal{D}$ for the locally homogeneous vector bundle E, which is defined by the spin representation τ_n of maximal compact subgroup $\mathrm{Spin}(2n+1)$. We introduce an operator $\mathcal{D}_0(i)$ naturally determined by $\mathcal{D}$, and using this we define the relative trace. In Section 3, we introduce the Selberg trace formula for nontrivial locally homogeneous vector bundles over noncompact locally symmetric spaces of rank 1. In Section 4, we use result from [9] to analyze the unipotent terms. In Section 5, we derive the relation between the relative trace and the spectral side of the Selberg trace formula. In Section 6, we define $\eta_\mathcal{D}(s)$ and $\zeta_{\mathcal{D}^2}(s)$ and prove Theorem 1.1. In Sections 7 and 8, we prove Theorem 1.2 and Theorem 1.3 using results of Sections 3, 4 and 5.

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2. Dirac operators on odd dimensional hyperbolic manifolds with cusps. Let G be a noncompact connected simple Lie group with finite center and let

$\mathfrak{g}$ denote its Lie algebra. We assume that the real rank of G is one. Let Γ be a discrete subgroup of G such that the group generated by the eigenvalues of Γ contains no root of unity. Let K be a maximal compact subgroup of G, θ the associated Cartan involution, $\mathfrak{k} \oplus \mathfrak{p}$ the associated Cartan decomposition, and $C(\ ,\ )$ the Killing form of $\mathfrak{g}$. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$; $\mathfrak{a}$ is one dimensional since the rank of G is 1. Let Φ be the set of roots of $(\mathfrak{g}, \mathfrak{a})$. Choose an order for Φ and let $\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\lambda \in \Phi^+} \mathfrak{g}_\lambda \oplus \sum_{\lambda \in \Phi^+} \mathfrak{g}_{-\lambda}$ be the root space decomposition where $\mathfrak{g}_0$ is the centralizer of $\mathfrak{a}$. Let α_1 be the unique simple positive root; then $\Phi^+ = \{\alpha_1, 2\alpha_1\}$. Let $\mathfrak{n}_{\alpha_1} := \mathfrak{g}_{\alpha_1}$ and $\mathfrak{n}_{2\alpha_1} := \mathfrak{g}_{2\alpha_1}$. This last space may be 0. If $N = \exp(\mathfrak{n}_{\alpha_1} \oplus \mathfrak{n}_{2\alpha_1})$ and $A = \exp(\mathfrak{a})$, then $G = NAK$ is an Iwasawa decomposition. Let $P_0 = NAM$ (with M the centralizer of A in K) be the associated minimal parabolic. The G-conjugates of P_0 are the proper parabolic subgroups of G. A parabolic subgroup P is called Γ-cuspidal if $\Gamma \cap N(P) \backslash N(P)$ is compact. Here $N(P)$ is the unipotent radical of P which, we may assume, is G-conjugate to N. Let $P_\Gamma = \{P_1, \ldots, P_\kappa\}$ be a complete set of Γ-conjugacy classes of Γ-cuspidal subgroups of G.

From now on, we assume that $G = \mathrm{Spin}(2n+1, 1)$ and $K = \mathrm{Spin}(2n+1)$. Let $\widetilde{X}$ denote the noncompact symmetric space given by $\mathrm{Spin}(2n+1,1)/\mathrm{Spin}(2n+1)$. The Killing form $C(\ ,\ )$ provides us with an invariant metric on G/K by $\langle Y, Z \rangle = \frac{1}{4n} C(Y, Z)$ for $Y, Z \in \mathfrak{p}$, which gives us constant curvature (-1). We use the spin representation (τ_n, V_{τ_n}) of $\mathrm{Spin}(2n+1)$ to define a homogeneous vector bundle $\widetilde{E}$ over $\widetilde{X} = \mathrm{Spin}(2n+1, 1)/\mathrm{Spin}(2n+1)$ by $\widetilde{E} = G \times V_{\tau_n}/\sim$ where

$$(g, v) \sim (g', v') \qquad \text{if} \quad (g', v') = (gk, \tau_n(k^{-1})v)$$

for $g, g' \in G$, $k \in K$. We denote such equivalence classes by $[g, v]$. Note that $\widetilde{E}$ admits a left G action defined by $g_0[g, v] = [g_0 g, v]$. If we restrict the spin representation τ_n of $K = \mathrm{Spin}(2n+1)$ to $M = \mathrm{Spin}(2n)$, then τ_n decomposes into two half spin representations σ_+, σ_- of $\mathrm{Spin}(2n)$. Two representations $(\sigma_+, H_{\sigma_+}), (\sigma_-, H_{\sigma_-})$ are unramified and $w\sigma_+ = \sigma_-, w\sigma_- = \sigma_+$ for the nontrivial element $w \in W(A) = M^*/M$ where M^* is the normalizer of A in K. We define the Dirac operator $\widetilde{D} : C^\infty(\widetilde{X}, \widetilde{E}) \rightarrow C^\infty(\widetilde{X}, \widetilde{E})$ by

$$\widetilde{D} = \sum_{i=1}^{2n+1} c(X_i) \nabla_{X_i},$$

where $\{X_i : 1 \leq i \leq 2n+1\}$ is a left invariant orthonormal frame such that $H := X_{2n+1}$ at eK spans $\mathfrak{a}$, and ∇ is the Levi-Civita connection on $\widetilde{E}$.

We consider a locally symmetric space given by $X = \Gamma \backslash \mathrm{Spin}(2n+1, 1)/\mathrm{Spin}(2n+1)$ where Γ is a discrete subgroup of $\mathrm{Spin}(2n+1, 1)$ with unipotent elements satisfying the condition in the introduction. As a consequence of this assumption, Γ is torsion free and $\Gamma \cap P = \Gamma \cap N(P)$ so that $\Gamma \cap P \backslash N(P) = \Gamma \cap N(P) \backslash N(P)$. Then X is a $(2n+1)$-dimensional hyperbolic manifold with cusps.

We denote by E the quotient space $\Gamma\backslash\widetilde{E}$. This is a locally homogeneous vector bundle over X. Moreover the Dirac operator $\widetilde{D}$ can be pushed down to X. We denote this operator by D and its unique self adjoint extension on $L^2(X,E)$ by $\mathcal{D}$. The hyperbolic manifold X endowed with the metric $\langle\ ,\ \rangle$ has a following decomposition,

$$X = X_0 \cup W_1 \cup \cdots \cup W_\kappa, \tag{1}$$

where X_0 is a compact manifold with boundaries and W_i, $i = 1, \ldots, \kappa$ are ends of X. (In general, X_0 and W_i's may have the nonempty intersections each other.) For each end W_i which we call as *cusp*, $W_i \cong [0,\infty) \times N_i$ and N_i can be identified with the flat torus T^{2n} with the metric dn^2, and the restriction of $\langle\ ,\ \rangle$ to W_i has the form $dg^2(r,x) = dr^2 + e^{-2r}dn^2(x)$ for $(r,x) \in [0,\infty) \times N_i$.

Now we have the following expression for D over the cusp W_i ,

$$D = c(H)(\nabla_H + B - n\,\mathrm{Id}) \tag{2}$$

where $B = \sum_{i=1}^{2n} c(X_i)c(H)\nabla^N_{X_i}$ with Levi-Civita connection ∇^N over $E|_N$ (see (7) in [2]). Note that B has the property $c(H)B = -Bc(H)$ and $E_0 := \ker(B)$ can be identified with V_{τ_n}. Let $0 < \mu_1 \le \mu_2 \le \mu_3 \le \cdots \to \infty$ be positive eigenvalues of B, each eigenvalue repeated according to its multiplicity with corresponding eigensections ψ_j. We denote by E_{μ_j} the eigenspace corresponding to μ_j. We decompose

$$L^2(\mathbb{R}^+ \times N_i, E|_{\mathbb{R}^+\times N_i}, dg^2(r,x))$$

into

$$L^2(\mathbb{R}^+, E_0, e^{-2nr}dr) \oplus \oplus_{j=1}^{\infty} L^2(\mathbb{R}^+, E_{\mu_j} \oplus c(H)E_{\mu_j}, e^{-2nr}dr)$$

where $\mathbb{R}^+$ denotes $[0,\infty)$. Note that $E_0 \cong V_{\tau_n}$ is a symplectic vector space with a symplectic structure $\langle c(H)\ \ ,\ \rangle$. We fix a Lagrangian subspace L of $E_0 \cong V_{\tau_n}$ such that $E_0 = L \oplus c(H)L$. Then the map

$$\alpha_j\psi_j + \alpha_{-j}c(H)\psi_j \to\ e^{-nr}\begin{pmatrix}\alpha_j\\ \alpha_{-j}\end{pmatrix}$$

gives a unitary equivalence

$$\begin{aligned}&L^2(\mathbb{R}^+, E_0, e^{-2nr}dr) \oplus \oplus_{j=1}^{\infty} L^2(\mathbb{R}^+, E_{\mu_j} \oplus c(H)E_{\mu_j}, e^{-2nr}dr)\\ &\quad\cong \oplus_{j=1}^{d(\sigma_+)} L^2(\mathbb{R}^+, \mathbb{C}^2, dr) \oplus \oplus_{j=1}^{\infty} L^2(\mathbb{R}^+, \mathbb{C}^2, dr)\end{aligned}$$

under which the Dirac operator D over the cusp W_i decomposes as follows

$$D \to \quad \oplus_{i=1}^{d(\sigma_+)} D_0 \oplus \oplus_{j=1}^{\infty} D_{\mu_j} \tag{3}$$

where

$$D_\mu = \begin{pmatrix} 0 & -\frac{d}{dr} + \mu \\ \frac{d}{dr} + \mu & 0 \end{pmatrix}.$$

Hence the Dirac Laplacian D^2 is transformed into $\oplus_{\mu \in \mathrm{spec}(B)} D_\mu^2$ where $D_\mu^2 = -\frac{d^2}{dr^2} + \mu^2$ on $L^2(\mathbb{R}^+, \mathbb{C}, dr)$.

We now consider the operator

$$c(H)\left(\frac{d}{dr} - n\,\mathrm{Id}\right) : C_0^\infty(\mathbb{R}^+, E_0) \to C_0^\infty(\mathbb{R}^+, E_0)$$

whose L^2-extension (with respect to $e^{-2nr}dr$) is transformed to $d(\sigma_+)$-copies of D_0 in (3). Now, we put

$$C_0^\infty(\mathbb{R}^+, E_0, L) := \{\phi \in C_0^\infty(\mathbb{R}^+, E_0) \mid \phi(0) \in L\},$$

then the following operator

$$c(H)\left(\frac{d}{dr} - n\,\mathrm{Id}\right) : C_0^\infty(\mathbb{R}^+, E_0, L) \;\to\; L^2(\mathbb{R}^+, E_0, e^{-2nr}dr) \tag{4}$$

is essentially self adjoint. By the natural embedding of $\mathbb{R}^+$ into the geodesic rays in $W_i \subset X$, we can regard $L^2(\mathbb{R}^+, E_0, e^{-2nr}dr)$ as a subspace of $L^2(X, E)$. The operator $c(H)(\frac{d}{dr} - n\,\mathrm{Id})$ in (4) can be extended to the self adjoint operator on $L^2(X, E)$ by the zero map over the orthogonal complement of this subspace in $L^2(X, E)$. For each W_i, in this way we obtain the operator $\mathcal{D}_0(i)$, $i = 1, \ldots, \kappa$. We can see that each $\mathcal{D}_0(i)$ has no point spectrum. Now we have:

PROPOSITION 2.1. *The differences* $\left(e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa} e^{-t\mathcal{D}_0^2(i)}\right)$, $\left(\mathcal{D}e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa} \mathcal{D}_0(i)\, e^{-t\mathcal{D}_0^2(i)}\right)$ *are trace class operators on* $L^2(X, E)$ *for* $t > 0$.

Proof. First choose a smooth function g_s such that $0 < g_s \le 1$, $g_s(x) = 1$ for $x \in X_0$ and $g_s((r, \cdot)) = 0$ for $(r, \cdot) \in W_i$ with $r \ge s$ for $1 \le i \le \kappa$. Denote by U_{g_s}, U_{1-g_s} the operators in $L^2(X, E)$ defined by multiplication by $g_s, 1 - g_s$ respectively. Then, for $s \gg 0$, $U_{g_s}(\sum_{i=1}^{\kappa} e^{-t\mathcal{D}_0^2(i)})$ is of trace class and the support of $U_{1-g_s}(\sum_{i=1}^{\kappa} e^{-t\mathcal{D}_0^2(i)})$ consists of disjoint κ-components lying in W_i's. So, for our proof we may assume that the supports of $e^{-t\mathcal{D}_0^2(i)}$, W_i's are disjoint each other. Now pick $f \in C^\infty(X)$ such that $0 < f \le 1$, $f(x) = 1$ for $x \in X_0$ and $f((r, \cdot)) = e^{-\frac{r}{4}}$ for $(r, \cdot) \in W_i$ with $r \gg 0$ for $1 \le i \le \kappa$. Denote by U_f the operator in $L^2(X,E)$ defined by multiplication by f. Then we may write

$$\begin{aligned} e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa} e^{-t\mathcal{D}_0^2(i)} &= (e^{-\frac{t}{2}\mathcal{D}^2} - \sum_{i=1}^{\kappa} e^{-\frac{t}{2}\mathcal{D}_0^2(i)}) \circ U_f^{-1} \circ U_f \circ e^{-\frac{t}{2}\mathcal{D}^2} \\ &\quad + \sum_{i=1}^{\kappa} e^{-\frac{t}{2}\mathcal{D}_0^2(i)} \circ U_f \circ U_f^{-1} \circ \left(e^{-\frac{t}{2}\mathcal{D}^2} - \sum_{i=1}^{\kappa} e^{-\frac{t}{2}\mathcal{D}_0^2(i)} \right). \end{aligned}$$

The heat kernel estimates for $e^{-t\mathcal{D}^2}$ and $\sum_{i=1}^{\kappa} e^{-t\mathcal{D}_0^2(i)}$ show that $(e^{-\frac{t}{2}\mathcal{D}^2} - \sum_{i=1}^{\kappa} e^{-\frac{t}{2}\mathcal{D}_0^2(i)}) \circ U_f^{-1}$, $U_f \circ e^{-\frac{t}{2}\mathcal{D}^2}$, $\sum_{i=1}^{\kappa} e^{-\frac{t}{2}\mathcal{D}_0^2(i)} \circ U_f$ and $U_f^{-1} \circ (e^{-\frac{t}{2}\mathcal{D}^2} - \sum_{i=1}^{\kappa} e^{-\frac{t}{2}\mathcal{D}_0^2(i)})$ are Hilbert-Schmidt operators. The composition of Hilbert-Schmidt operators is of trace class, hence $e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa} e^{-t\mathcal{D}_0^2(i)}$ is a trace class operator. The remaining case is proved in the same way. □

3. The Selberg trace formula. Let R_Γ be the right regular representation of G on $L^2(\Gamma\backslash G)$ and f be a right K-finite function in the Harish-Chandra L^p-Schwartz space $\mathcal{C}^p(G)$ where $0 < p < 1$. The trace of the restriction $R_\Gamma(f)$ to the discrete part $L^2_d(\Gamma\backslash G)$ of $L^2(\Gamma\backslash G)$ can be written as

$$\operatorname{Tr}(R_\Gamma(f)|_{L^2_d(\Gamma\backslash G)}) = I_\Gamma(f) + H_\Gamma(f) + U_\Gamma(f) + S_\Gamma(f) + T_\Gamma(f).$$

Here

$$\begin{aligned} I_\Gamma(f) &= \operatorname{Vol}(\Gamma\backslash G) f(1), \\ H_\Gamma(f) &= \sum_{\{\gamma:\text{hyperbolic}\}} \operatorname{Vol}(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)dx, \\ U_\Gamma(f) &= \frac{1}{|\alpha_1|}\{C_1(\Gamma)T_1(f) + C_2(\Gamma)T_2(f) + C_1'(\Gamma)T_1'(f)\}, \\ S_\Gamma(f) &= \frac{1}{4\pi i}\sum_{\sigma\in\hat{M}} \int_{\operatorname{Re}(\nu)=0} \operatorname{Tr}\left(C_\Gamma(\sigma,\nu)^{-1}\frac{d}{d\nu}C_\Gamma(\sigma,\nu)\pi_\Gamma(\sigma,\nu)(f)\right) d\nu, \\ T_\Gamma(f) &= -\frac{1}{4}\sum_{\sigma\in\hat{M}} \operatorname{Tr}(C_\Gamma(\sigma,0)\pi_\Gamma(\sigma,0)(f)) \end{aligned}$$

where Γ_γ, G_γ are centralizers of γ in Γ, G respectively, the constants $C_1(\Gamma)$, $C_2(\Gamma)$, $C_1'(\Gamma)$ and tempered distributions T_1, T_2, T_1' will be discussed in the next section, and

$$C_\Gamma(\sigma,\nu) : \mathcal{H}_\Gamma(\sigma,\nu) \to \mathcal{H}_\Gamma(\sigma,-\nu)$$

denotes the intertwining operator (see pp. 166–167 in [1], pp. 9–10 in [22] for the precise definitions of $C_\Gamma(\sigma,\nu), \mathcal{H}_\Gamma(\sigma,\nu)$). We refer to [1], [20], [22], [23] for detailed expositions of the Selberg trace formula. The various invariant measures that we use are normalized as in [20]. More precisely, let d_K be the Haar measure on K which assigns to K total volume one, let d_A, d_N be the exponentiation of the normalized Lebesgue measure on the Lie algebra $\mathfrak{a}$, $\mathfrak{n}$ of A, N, respectively, relative to the Euclidean structure associated with the Killing form. Then the Haar measure d_G is determined by

$$d_G(x) = a^{2\rho}\, d_N(n)\, d_A(a)\, d_K(k) \quad \text{for} \quad x = nak$$

where ρ is the sum of the positive roots of $(\mathfrak{g}, \mathfrak{a})$ divided by 2.

The kernel $\widetilde{K}_t(x : y)$ of the integral operator $e^{-t\widetilde{D}^2}$ (or $\widetilde{D}e^{-t\widetilde{D}^2}$) over G/K is a section of $\widetilde{E} \boxtimes \widetilde{E}^*$, the external tensor product of $\widetilde{E}$ and $\widetilde{E}^*$ over $G \times G$. The bundle E is a trivial bundle, hence the kernel $\widetilde{K}_t(x : y)$ is an element of $(C^\infty(G \times G) \otimes \operatorname{End}(V_{\tau_n}))^K$, which consists of endomorphism valued functions on $G \times G$ invariant under K. It follows that there exists a function $\widetilde{K}_t(x) : G \to \operatorname{End}(V_{\tau_n})$ such that

$$\widetilde{K}_t(x : y) = \widetilde{K}_t(x^{-1}y) \quad \text{and} \quad \widetilde{K}_t(k_1^{-1}xk_2) = \tau_n(k_1^{-1})\widetilde{K}_t(x)\tau_n(k_2)$$

for $x, y \in G, k_1, k_2 \in K$. We define the local trace of $\widetilde{K}_t(x)$ to be the scalar function on G given by $K_t(x) := \operatorname{tr}(\widetilde{K}_t(x))$. We denote by $\widehat{K}_t(\sigma, \nu)$ the Fourier transform of K_t for the unitary principal representation $\pi_{\sigma,\nu}$ of G, that is,

$$\widehat{K}_t(\sigma, \nu) := \operatorname{Tr}(\pi_{\sigma,\nu}(K_t)).$$

From now on, we will denote by K_t^e, K_t^o the scalar functions corresponding to $e^{-t\widetilde{D}^2}$, $\widetilde{D}e^{-t\widetilde{D}^2}$. By (4.5) in [13], we have:

PROPOSITION 3.1. *For $\lambda \in \mathbb{R}$, we have*

$$\widehat{K}_t^e(\sigma_\pm, i\lambda) = e^{-t\lambda^2}, \quad \widehat{K}_t^o(\sigma_+, i\lambda) = \lambda e^{-t\lambda^2}, \quad \widehat{K}_t^o(\sigma_-, i\lambda) = -\lambda e^{-t\lambda^2}.$$

Since $\sigma_\pm$ is unramified, the intertwining operator $C_\Gamma(\sigma_+, \nu) = C_\Gamma(\sigma_-, \nu)$ acting on $\mathcal{H}_\Gamma(\sigma_+, \nu) = \mathcal{H}_\Gamma(\sigma_-, \nu)$ switches the subspaces induced by the representations σ_+, σ_-. Hence, $C_\Gamma(\sigma_+, \nu)$ takes the form

$$C_\Gamma(\sigma_+, \nu) = \begin{pmatrix} 0 & C_-(\nu) \\ C_+(\nu) & 0 \end{pmatrix}$$

with respect to the decomposition of $\mathcal{H}_\Gamma(\sigma_+, \nu)$. Therefore, we have

$$C_\Gamma(\sigma_+, \nu)^{-1} C'_\Gamma(\sigma_+, \nu) = \begin{pmatrix} C_-(-\nu)C'_+(\nu) & 0 \\ 0 & C_+(-\nu)C'_-(\nu) \end{pmatrix}$$

and

$$T_\Gamma(\cdot) = -\frac{1}{4}\mathrm{Tr}(C_\Gamma(\sigma_+, 0)\pi_\Gamma(\sigma_+, 0)(\cdot)) = 0.$$

Since $\tau_n|_M = \sigma_+ \oplus \sigma_-$, the Selberg trace formulas applied to K_t^e, K_t^o are given by

$$\begin{aligned}
(5)\quad \sum_{\sigma=\sigma_\pm}\sum_{\lambda_k\in\sigma_p^\pm} \widehat{K}_t^e(\sigma, i\lambda_k) - \frac{1}{4\pi}\int_{-\infty}^{\infty} \mathrm{Tr}(C_\Gamma(\sigma_+, -i\lambda)C_\Gamma'(\sigma_+, i\lambda)\pi_\Gamma(\sigma_+, i\lambda)(K_t^e))\, d\lambda \\
= \sum_{\lambda_k\in\sigma_p} e^{-t\lambda_k^2} - \frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty} e^{-t\lambda^2}\mathrm{tr}(C_\Gamma(\sigma_+, -i\lambda)C_\Gamma'(\sigma_+, i\lambda))\, d\lambda \\
= I_\Gamma(K_t^e) + H_\Gamma(K_t^e) + U_\Gamma(K_t^e),
\end{aligned}$$

$$\begin{aligned}
(6)\quad \sum_{\sigma=\sigma_\pm}\sum_{\lambda_k\in\sigma_p^\pm} \widehat{K}_t^o(\sigma, i\lambda_k) - \frac{1}{4\pi}\int_{-\infty}^{\infty} \mathrm{Tr}(C_\Gamma(\sigma_+, -i\lambda)C_\Gamma'(\sigma_+, i\lambda)\pi_\Gamma(\sigma_+, i\lambda)(K_t^o))d\lambda \\
= \sum_{\lambda_k\in\sigma_p} \lambda_k e^{-t\lambda_k^2} - \frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty} \lambda e^{-t\lambda^2}\mathrm{tr}(C_-(-i\lambda)C_+'(i\lambda))\, d\lambda \\
+ \frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty} \lambda e^{-t\lambda^2}\mathrm{tr}(C_+(-i\lambda)C_-'(i\lambda))\, d\lambda \\
= \sum_{\lambda_k\in\sigma_p} \lambda_k e^{-t\lambda_k^2} - \frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty} \lambda e^{-t\lambda^2}\mathrm{tr}_s(C_\Gamma(\sigma_+, -i\lambda)C_\Gamma'(\sigma_+, i\lambda))\, d\lambda \\
= I_\Gamma(K_t^o) + H_\Gamma(K_t^o) + U_\Gamma(K_t^o),
\end{aligned}$$

where $d(\sigma_+)$ is the degree of σ_+ (although we know $d(\sigma_+) = 2^{n-1}$, we will use the notation $d(\sigma_+)$ instead of 2^{n-1} since this indicates the origin of the constant factor) and

$$\mathrm{tr}_s(C_\Gamma(\sigma_+, -\nu)C_\Gamma'(\sigma_+, \nu)) := \mathrm{tr}(C_-(-\nu)C_+'(\nu)) - \mathrm{tr}(C_+(-\nu)C_-'(\nu)).$$

For $I_\Gamma(K_t)$, we have

$$I_\Gamma(K_t) = \mathrm{Vol}(\Gamma\backslash G)\left(\int_{-\infty}^{\infty} \widehat{K}_t(\sigma_+, i\lambda)p(\sigma_+, i\lambda)\, d\lambda + \int_{-\infty}^{\infty} \widehat{K}_t(\sigma_-, i\lambda)p(\sigma_-, i\lambda)\, d\lambda\right),$$

where $p(\sigma_\pm, i\lambda)$ is the Plancherel measure which is an even polynomial with respect to λ. The equalities $\widehat{K}_t^e(\sigma_\pm, i\lambda) = e^{-t\lambda^2}$ and $p(\sigma_+, i\lambda) = p(\sigma_-, i\lambda)$ give

$$(7)\qquad I_\Gamma(K_t^e) = 2\,\mathrm{Vol}(\Gamma\backslash G)\int_{-\infty}^{\infty} e^{-t\lambda^2}p(\sigma_+, i\lambda)\, d\lambda$$

and $\widehat{K}_t^o$ is odd with respect to λ so that

$$I_\Gamma(K_t^o) = 0. \tag{8}$$

It is well known that

$$\begin{aligned} H_\Gamma(K_t^e) &= \frac{1}{\sqrt{4\pi t}} \\ &\times \sum_{\gamma:\text{hyperbolic}} l(C_\gamma) j(\gamma)^{-1} D(\gamma)^{-1} \left(\overline{\chi_{\sigma_+}(m_\gamma)} + \overline{\chi_{\sigma_-}(m_\gamma)}\right) e^{-\frac{l(C_\gamma)^2}{4t}}, \end{aligned} \tag{9}$$

$$\begin{aligned} H_\Gamma(K_t^o) &= \frac{2\pi i}{(4\pi t)^{\frac{3}{2}}} \\ &\times \sum_{\gamma:\text{hyperbolic}} l(C_\gamma)^2 j(\gamma)^{-1} D(\gamma)^{-1} \left(\overline{\chi_{\sigma_+}(m_\gamma)} - \overline{\chi_{\sigma_-}(m_\gamma)}\right) e^{-\frac{l(C_\gamma)^2}{4t}}, \end{aligned} \tag{10}$$

where $l(C_\gamma)$ is the length of the closed geodesic C_γ, $j(\gamma)$ is the positive integer such that $\gamma = \gamma_0^{j(\gamma)}$ for a primitive element γ_0, $D(\gamma) = e^{nl(C_\gamma)}|\det(\operatorname{Ad}(a_\gamma m_\gamma)^{-1} - I|_{\mathfrak{n}})|$ for the element $a_\gamma m_\gamma \in A^+M$ which is conjugate to γ and χ_σ is the character of σ.

4. Unipotent terms. In this section we compute unipotent terms $U_\Gamma(K_t^e)$, $U_\Gamma(K_t^o)$. We employ the formula obtained by Hoffmann (see [9]) to compute these terms explicitly. By this explicit computation and Proposition 3.1, it follows that $U_\Gamma(K_t^o) = 0$. This simplifies many steps related to the application of the Selberg trace formula for K_t^o.

For a real rank 1 group G, the unipotent term for a right K-finite function f in $\mathcal{C}^p(G)$, $0 < p < 1$ is given by

$$U_\Gamma(f) = \frac{1}{|\alpha_1|}\{C_1(\Gamma)T_1(f) + C_2(\Gamma)T_2(f) + C_1'(\Gamma)T_1'(f)\}, \tag{11}$$

(see theorem on p. 299 of [20]). Here α_1 is the unique simple positive root for $(\mathfrak{g}, \mathfrak{a})$, the constants $C_1(\Gamma), C_2(\Gamma), C_1'(\Gamma)$ which depend on Γ are computed in [3] and

$$\begin{aligned} T_1(f) &= \frac{1}{A(\mathfrak{n}_1)} \int_N \int_K f(k^{-1}nk)\, dk dn \\ T_2(f) &= \frac{|\alpha_1|}{2} \left\{ \int_{G/G_{n_0}} f(xn_0x^{-1})\, dx + \int_{G/G_{n_0^{-1}}} f(xn_0^{-1}x^{-1})\, dx \right\} \\ T_1'(f) &= \frac{m_1 + 2m_2}{A(\mathfrak{n}_1)} \int_{N_1} \int_{N_2} \int_K f(k^{-1}n_1n_2k) \log|\log(n_1)|\, dk dn_1 dn_2, \end{aligned}$$

where n_0 is a representative of the nontrivial unipotent orbit in $\mathfrak{n}_2$, $m_i = \dim(\mathfrak{n}_i)$ and $A(\mathfrak{n}_1)$ is the volume of the unit sphere in $\mathfrak{n}_1$ and G_n is the centralizer of an element n in G.

In our case, $G = \mathrm{Spin}(2n+1,1)$, the second term $T_2(f) = 0$ and $m_2 = 0$ in the third term $T_1'(f)$ since $N = N_1$. The first term T_1 for K_t is given by

$$T_1(K_t) = \frac{1}{2\pi A(\mathfrak{n}_1)} \left(\int_{-\infty}^{\infty} \widehat{K}_t(\sigma_+, i\lambda)\, d\lambda + \int_{-\infty}^{\infty} \widehat{K}_t(\sigma_-, i\lambda)\, d\lambda \right).$$

We have $\widehat{K}_t^e(\sigma_\pm, i\lambda) = e^{-t\lambda^2}$ and $\widehat{K}_t^o(\sigma_\pm, i\lambda) = \pm\lambda e^{-t\lambda^2}$, which implies

$$T_1(K_t^e) = \frac{1}{\pi A(\mathfrak{n}_1)} \int_{-\infty}^{\infty} e^{-t\lambda^2}\, d\lambda \quad \text{and} \quad T_1(K_t^o) = 0. \tag{12}$$

The third term $T_1'(f)$ is more complicated and we need to introduce some notation. Let T_M be a Cartan subgroup in M, so that $T = A \cdot T_M$ is a Cartan subgroup of G. Let Σ_M denote the set of positive roots for $(\mathfrak{m}_\mathbb{C}, \mathfrak{t}_{\mathfrak{m}\mathbb{C}})$. Let ρ_{Σ_M} be the half sum of elements in Σ_M. We denote by Σ_A the set of positive roots of $(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ which do not vanish on $\mathfrak{a}_\mathbb{C}$. The union of Σ_M with Σ_A gives the set of positive roots for $(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ denoted by Σ_G. Let $H_\alpha \in \mathfrak{t}_\mathbb{C}$ be the coroot corresponding to $\alpha \in \pm\Sigma_G$, that is, $\alpha(H_\alpha) = 2, \alpha'(H_\alpha) \in \mathbb{Z}$ for all $\alpha, \alpha' \in \pm\Sigma_G$ and $\Pi := \prod_{\alpha \in \Sigma_M} H_\alpha$ is an element of the symmetric algebra $S(\mathfrak{t}_{\mathfrak{m}\mathbb{C}})$. We denote the simple reflection corresponding to $\alpha \in \Sigma_G$ by s_α. Following the corollary on p. 96 of [9], we put

$$I(f) := \frac{1}{2\pi} \sum_{\sigma \in \hat{M}} \int_{-\infty}^{\infty} \Omega(\sigma, -i\lambda) \hat{f}(\sigma, i\lambda)\, d\lambda$$

where

$$\begin{aligned} \Omega(\sigma, i\lambda) \;=\; & 2d(\sigma)\psi(1) - \frac{1}{2} \sum_{\alpha \in \Sigma_A} \lambda(H_\alpha) \frac{\Pi(s_\alpha \lambda_\sigma)}{\Pi(\rho_{\Sigma_M})} \\ & \times \Big(\psi(1 + \lambda_\sigma(H_\alpha)) + \psi(1 - \lambda_\sigma(H_\alpha)) \Big), \end{aligned}$$

and where $d(\sigma)$ is the degree of σ, ψ is the logarithmic derivative of the Gamma function and $\lambda_\sigma - \rho_{\Sigma_M}$ is the highest weight of $(\sigma, i\lambda) \in \hat{M} \times i\mathfrak{a}$.

The principal series representation $(\pi_{\sigma,\nu}, \mathcal{H}_{\sigma,\nu})$ depends on the parabolic subgroup P and we denote its dependence on P by $(\pi_{\sigma,\nu}(P), \mathcal{H}_{\sigma,\nu}(P))$. The intertwining operator

$$J_{\bar{P}|P}(\sigma, \nu) : \mathcal{H}_{\sigma,\nu}(P) \to \mathcal{H}_{\sigma,\nu}(\bar{P})$$

is defined by

$$(J_{\bar{P}|P}(\sigma,\nu)\phi) = \int_{\bar{N}} \phi(x\bar{n})\, d\bar{n}$$

and satisfies

$$J_{\bar{P}|P}(\sigma,\nu)\pi_{\sigma,\nu}(P) = \pi_{\sigma,\nu}(\bar{P})J_{\bar{P}|P}(\sigma,\nu).$$

The restriction to K defines an isomorphism from $\mathcal{H}_{\sigma,\nu}(P)$ to $\mathcal{H}_\sigma(P)$. Then $J_{\bar{P}|P}(\sigma,\nu)$ can be considered as a family of operators from $\mathcal{H}_\sigma(P)$ to $\mathcal{H}_\sigma(\bar{P})$. Here $\mathcal{H}_\sigma(P)$ is the space of all measurable functions $v : K \longrightarrow H_\sigma$ such that

$$v(km) = \sigma(m)^{-1} v(k)$$

for all $m \in M$, $k \in K$. Let

$$J_P(\sigma,\nu : f) := -\mathrm{Tr}(\pi_{\sigma,\nu}(f)J_{\bar{P}|P}(\sigma,\nu)^{-1}\partial_\nu J_{\bar{P}|P}(\sigma,\nu)),$$

where the derivative ∂_ν is taken with respect to ν for the family of operators $J_{\bar{P}|P}(\sigma,\nu)$ acting on $\mathcal{H}_\sigma(P)$. Then there exists the Harish Chandra C-function $C_\tau(\sigma,\nu)$ such that

$$T_\tau J_{\bar{P}|P}(\sigma,\nu)^{-1}\partial_\nu J_{\bar{P}|P}(\sigma,\nu) \;=\; C_\tau(\sigma,\nu)^{-1}\partial_\nu C_\tau(\sigma,\nu)T_\tau,$$

where T_τ is the projection to the τ-isotypic component of $\mathcal{H}_{\sigma,\nu}(P)$. We refer to [6] for more detail. We have the following proposition for $C_{\tau_n}(\sigma_\pm,\nu)$:

Proposition 4.1. *For the half spin representation* $\sigma_\pm$ *of* Spin$(2n)$,

$$C_{\tau_n}(\sigma_+, i\lambda) = C_{\tau_n}(\sigma_-, i\lambda) = \frac{(2n-1)!}{(n-1)!}\frac{\Gamma(i\lambda+\frac{1}{2})}{\Gamma(i\lambda+n+\frac{1}{2})}\mathrm{Id}.$$

Proof. This follows from Theorem 8.2 in [6]. □

By the equality (8) (see also (49)) in [9], the weighted orbital integral T_1' is given by

$$T_1'(f)=\frac{m_1}{A(\mathfrak{n}_1)}\Big(I(f)+\frac{1}{2\pi}\text{ p.v.}\sum_{\sigma\in\hat{M}} d(\sigma)\int_{-\infty}^{\infty} J_P(\sigma,i\lambda : f)\, d\lambda+\sum_{\sigma\in\hat{M}} d(\sigma)\frac{n(\sigma)}{2}\hat{f}(\sigma,0)\Big),$$

where p.v. means the Cauchy principal value and $2n(\sigma)$ is the order of the zero of $p(\sigma,\nu)$ at $\nu=0$. Using that $n(\sigma_\pm)=0$ and Proposition 4.1 we obtain

$$T_1'(K_t^e) = \frac{m_1}{2\pi A(\mathfrak{n}_1)} \times \int_{-\infty}^{\infty} e^{-t\lambda^2}\Big(\Omega(\sigma_+,-i\lambda)+\Omega(\sigma_-,-i\lambda)-2d(\sigma_+)\partial_{i\lambda}\log C_{\tau_n}(\sigma_+,i\lambda)\Big)\,d\lambda,$$

$$T_1'(K_t^o) = \frac{m_1}{2\pi A(\mathfrak{n}_1)}\int_{-\infty}^{\infty}\lambda e^{-t\lambda^2}\Big(\Omega(\sigma_+,-i\lambda)-\Omega(\sigma_-,-i\lambda)\Big)\,d\lambda.$$

Now we consider $\Omega(\sigma_\pm,i\lambda)$ in the following proposition.

PROPOSITION 4.2. *For the half spin representation $\sigma_\pm$ of* Spin$(2n)$, *we have*

$$\Omega(\sigma_+,i\lambda) = \Omega(\sigma_-,i\lambda) = -\frac{d(\sigma_+)}{2}\left(\psi\left(i\lambda-n+\frac{1}{2}\right)\right. \left.+\psi\left(-i\lambda-n+\frac{1}{2}\right)+\psi\left(i\lambda+\frac{1}{2}\right)+\psi\left(-i\lambda+\frac{1}{2}\right)\right)+P^n(\lambda)$$

where $P^n(\lambda)$ is an even polynomial of degree $(2n-4)$ for $n\geq 2$ and $P^1(\lambda)$ is a constant.

Proof. The $n=1$ case can be computed in the same way as for the $n\geq 2$ cases, hence we may assume that $n\geq 2$ in the following proof. The highest weight of the half spin representation $\sigma_\pm$ of Spin$(2n)\subset$ Spin$(2n+1)$ is given by

$$\frac{1}{2}(e_2+e_3+\cdots+e_n\pm e_{n+1})$$

with respect to the standard basis $\{e_i\}$. This implies that

$$i\lambda e_1+\sigma_\pm+\rho_{\Sigma_M} = i\lambda e_1+\left(n-\frac{1}{2}\right)e_2+\left(n-\frac{3}{2}\right)e_3+\cdots+\frac{3}{2}e_n\pm\frac{1}{2}e_{n+1}.$$

The positive roots $\alpha\in\Sigma_A$ are given by e_1-e_j, e_1+e_j for $2\leq j\leq n+1$. Then we can see that $s_{(e_1-e_j)}(i\lambda e_1+\sigma_\pm+\rho_{\Sigma_M})$ is

$$i\lambda e_j + \left(n-\frac{1}{2}\right)e_2+\cdots+(n-j+\frac{3}{2})e_1+\cdots+\frac{3}{2}e_n\pm\frac{1}{2}e_{n+1}\quad \text{if}\quad 2\leq j\leq n,$$

$$i\lambda e_{n+1} + (n-\frac{1}{2})e_2+\cdots+\frac{3}{2}e_n\pm\frac{1}{2}e_1\quad \text{if}\quad j=n+1$$

and $s_{(e_1+e_j)}(i\lambda e_1+\sigma_\pm+\rho_{\Sigma_M})$ is

$$-\,i\lambda e_j+\left(n-\frac{1}{2}\right)e_2+\cdots-\left(n-j+\frac{3}{2}\right)e_1+\cdots+\frac{3}{2}e_n\pm\frac{1}{2}e_{n+1}\quad \text{if}\quad 2\leq j\leq n,$$

$$-\,i\lambda e_{n+1}+\left(n-\frac{1}{2}\right)e_2+\cdots+\frac{3}{2}e_n\mp\frac{1}{2}e_1\quad \text{if}\quad j=n+1.$$

These give us the formula

$$
\begin{aligned}
(13)\quad & \Pi(s_{(e_1\pm e_j)}(i\lambda e_1+\sigma_\pm+\rho_{\Sigma_M})) \\
&= C_j\left(\lambda^2+\left(n-\frac{1}{2}\right)^2\right)\cdot\left(\lambda^2+\left(n-\frac{3}{2}\right)^2\right)\cdots\left(\lambda^2+\left(n-j+\frac{5}{2}\right)^2\right) \\
&\quad\cdot\left(-\lambda^2-\left(n-j+\frac{1}{2}\right)^2\right)\cdots\left(-\lambda^2-\left(\frac{3}{2}\right)^2\right)\cdot\left(-\lambda^2-\left(\frac{1}{2}\right)^2\right)
\end{aligned}
$$

where

$$
\begin{aligned}
C_j &= \prod_{1\le k<l\le n, k\ne j, l\ne j}\left(\left(n-k+\frac{1}{2}\right)^2-\left(n-l+\frac{1}{2}\right)^2\right) \\
&= \prod_{1\le k<l\le n, k\ne j, l\ne j}(l-k)(2n-k-l+1)
\end{aligned}
$$

for $2\le j\le n+1$. In particular, we obtain

$$
\begin{aligned}
\Pi(s_{e_1\pm e_j}(i\lambda e_1+\sigma_++\rho_{\Sigma_M})) &= \Pi(s_{e_1\pm e_j}(i\lambda e_1+\sigma_-+\rho_{\Sigma_M})) \\
\Pi(s_{e_1-e_j}(i\lambda e_1+\sigma_\pm+\rho_{\Sigma_M})) &= \Pi(s_{e_1+e_j}(i\lambda e_1+\sigma_\pm+\rho_{\Sigma_M}))
\end{aligned}
$$

for $2\le j\le n+1$. From now on, we denote by $P_j(\lambda)$ the polynomial of λ in (13) for $2\le j\le n+1$. Note that $P_j(\lambda)$ is an even polynomial of degree $2(n-1)$. On the other hand, $(i\lambda e_1+\sigma_\pm+\rho_{\Sigma_M})(H_\alpha)$ is given by

$$
\begin{aligned}
& i\lambda - \left(n-j+\frac{3}{2}\right) \quad \text{if}\quad \alpha=e_1-e_j,\quad 2\le j\le n \\
& i\lambda \mp \frac{1}{2} \quad \text{if}\quad \alpha=e_1-e_{n+1} \\
& i\lambda + \left(n-j+\frac{3}{2}\right) \quad \text{if}\quad \alpha=e_1+e_j,\quad 2\le j\le n \\
& i\lambda \pm \frac{1}{2} \quad \text{if}\quad \alpha=e_1+e_{n+1}.
\end{aligned}
$$

Then the pair $\left(\psi(1+(i\lambda e_1+\sigma_\pm+\rho_{\Sigma_M})(H_\alpha)), \psi(1-(i\lambda e_1+\sigma_\pm+\rho_{\Sigma_M})(H_\alpha))\right)$ is given by

$$
\begin{aligned}
& \psi\left(i\lambda-n+j-\frac{1}{2}\right),\ \psi\left(-i\lambda+n-j+\frac{5}{2}\right) \quad \text{for}\quad e_1-e_j,\quad 2\le j\le n \\
& \psi\left(i\lambda+\frac{1}{2}\right),\ \psi\left(-i\lambda+\frac{3}{2}\right) \quad \text{for}\quad e_1-e_{n+1},\quad \sigma=\sigma_+
\end{aligned}
$$

$$\psi\left(i\lambda+\frac{3}{2}\right),\quad \psi\left(-i\lambda+\frac{1}{2}\right)\qquad \text{for}\quad e_1-e_{n+1},\quad \sigma=\sigma_-$$

$$\psi\left(i\lambda+n-j+\frac{5}{2}\right),\quad \psi\left(-i\lambda-n+j-\frac{1}{2}\right)\qquad \text{for}\quad e_1+e_j,\quad 2\le j\le n$$

$$\psi\left(i\lambda+\frac{3}{2}\right),\quad \psi\left(-i\lambda+\frac{1}{2}\right)\qquad \text{for}\quad e_1+e_{n+1},\quad \sigma=\sigma_+$$

$$\psi\left(i\lambda+\frac{1}{2}\right),\quad \psi\left(-i\lambda+\frac{3}{2}\right)\qquad \text{for}\quad e_1+e_{n+1},\quad \sigma=\sigma_-.$$

Comparing the two sets

$$\left\{\Pi(s_\alpha(i\lambda e_1+\sigma_++\rho_{\Sigma_M})),1+(i\lambda e_1+\sigma_++\rho_{\Sigma_M})(H_\alpha),1-(i\lambda e_1+\sigma_++\rho_{\Sigma_M})(H_\alpha)\right\},$$

$$\left\{\Pi(s_\alpha(i\lambda e_1+\sigma_-+\rho_{\Sigma_M})),1+(i\lambda e_1+\sigma_-+\rho_{\Sigma_M})(H_\alpha),1-(i\lambda e_1+\sigma_-+\rho_{\Sigma_M})(H_\alpha)\right\}$$

we see that they are equal to each other, so that $\Omega(\sigma_+,i\lambda)=\Omega(\sigma_-,i\lambda)$. We now compute the exact form of $\Omega(\sigma_+,i\lambda)=\Omega(\sigma_-,i\lambda)$. Using the relations $\psi(z+1)=\frac{1}{z}+\psi(z)$, we have

$$\begin{aligned}\psi\left(i\lambda-n+j-\frac{1}{2}\right)&+\psi\left(-i\lambda-n+j-\frac{1}{2}\right)+\psi\left(i\lambda+n-j+\frac{5}{2}\right)+\psi\left(-i\lambda+n-j+\frac{5}{2}\right)\\ &=\frac{1}{\lambda^2+\left(\frac{1}{2}\right)^2}+\cdots+\frac{2\left(n-j+\frac{1}{2}\right)}{\lambda^2+\left(n-j+\frac{1}{2}\right)^2}+\frac{-2\left(n-j+\frac{5}{2}\right)}{\lambda^2+\left(n-j+\frac{5}{2}\right)^2}+\cdots+\frac{-2\left(n-\frac{1}{2}\right)}{\lambda^2+\left(n-\frac{1}{2}\right)^2}\\ &\quad+\psi\left(i\lambda-n+\frac{1}{2}\right)+\psi\left(-i\lambda-n+\frac{1}{2}\right)+\psi\left(i\lambda+\frac{1}{2}\right)+\psi\left(-i\lambda+\frac{1}{2}\right).\end{aligned}$$

Now using the formula (see the last line of p. 95 in [9]),

$$\sum_{\alpha\in\Sigma_A}\Pi(s_\alpha\lambda_\sigma)=2\Pi(\lambda_\sigma),$$

we decompose

$$\frac{1}{2}\sum_{\alpha\in\Sigma_A}\frac{\Pi(s_\alpha\lambda_\sigma)}{\Pi(\rho_{\Sigma_M})}\times\Big(\psi(1+\lambda_\sigma(H_\alpha))+\psi(1-\lambda_\sigma(H_\alpha))\Big)$$

into

$$\frac{d(\sigma_\pm)}{2}\left(\psi\left(i\lambda-n+\frac{1}{2}\right)+\psi\left(-i\lambda-n+\frac{1}{2}\right)+\psi\left(i\lambda+\frac{1}{2}\right)+\psi\left(-i\lambda+\frac{1}{2}\right)\right)$$

and

$$\frac{1}{4\Pi(\rho_{\Sigma_M})}\sum_{j=2}^{n+1} P_j(\lambda)R_j(\lambda)$$

where

$$R_j(\lambda) = \frac{1}{\lambda^2+(\frac{1}{2})^2}+\cdots+\frac{2(n-j+\frac{1}{2})}{\lambda^2+(n-j+\frac{1}{2})^2}+\frac{-2(n-j+\frac{5}{2})}{\lambda^2+(n-j+\frac{5}{2})^2}+\cdots+\frac{-2(n-\frac{1}{2})}{\lambda^2+(n-\frac{1}{2})^2}.$$

We can see, from the definitions of $P_j(\lambda)$ and $R_j(\lambda)$, that $P_j(\lambda)R_j(\lambda)$ is an even polynomial of degree $2n-4$. This ends the proof. □

For the constants in (11), comparing proposition 6.2 in [1] with theorem 2 in [3], we obtain

$$\frac{C_1'(\Gamma)}{|\alpha_1|}\frac{m_1}{A(\mathfrak{n}_1)} = \frac{\kappa}{2}. \tag{14}$$

Now we have the following corollary.

COROLLARY 4.3.

$$U_\Gamma(K_t^o) = 0, \qquad U_\Gamma(K_t^e) = \frac{2}{2\pi}\int_{-\infty}^{\infty} e^{-t\lambda^2}(P_U(\lambda)+Q(\lambda))\,d\lambda$$

where $P_U(\lambda)$ is an even polynomial of degree $(2n-4)$ and

$$Q(\lambda) = -\kappa\frac{d(\sigma_+)}{2}\left(\psi\left(i\lambda+\frac{1}{2}\right)+\psi\left(-i\lambda+\frac{1}{2}\right)\right).$$

Proof. The first claim follows easily from (12), (14), Proposition 4.1 and 4.2. For the second claim, $Q(\lambda)$ is *a priori* given by

$$-\frac{\kappa d(\sigma_+)}{2}\left(\frac{1}{2}\left(\psi\left(i\lambda-n+\frac{1}{2}\right)+\psi\left(-i\lambda-n+\frac{1}{2}\right)\right)\right.$$
$$\left.-\frac{1}{2}\left(\psi\left(i\lambda+n+\frac{1}{2}\right)+\psi\left(-i\lambda+n+\frac{1}{2}\right)\right)+\left(\psi\left(i\lambda+\frac{1}{2}\right)+\psi\left(-i\lambda+\frac{1}{2}\right)\right)\right).$$

If we use the relation $\psi(z+1)=\frac{1}{z}+\psi(z)$, then we can reduce the above formula to the claimed one for $Q(\lambda)$. □

5. Relative Traces and Spectral Sides. In this section we study relations of the relative traces with the spectral sides of the Selberg trace formulas applied to the test functions K_t^e, K_t^o. A formula of this type was proved by Müller for the similar cases in [15], [18]. Following [15], [18], we prove the corresponding formula for Dirac operators acting on spinor bundles over hyperbolic manifolds with cusps.

First, let us observe that $L^2(X, E)$ can be identified with the space

$$(L^2(\Gamma\backslash G) \otimes V_{\tau_n})^K = \{ f \in L^2(\Gamma\backslash G) \otimes V_{\tau_n} | f(xk) = \tau_n(k)^{-1} f(x) \quad \text{for} \quad k \in K, x \in G\}.$$

The decomposition of $L^2(\Gamma\backslash G) = L^2_d(\Gamma\backslash G) \oplus L^2_c(\Gamma\backslash G)$ allows us to decompose

$$(L^2(\Gamma\backslash G) \otimes V_{\tau_n})^K = (L^2_d(\Gamma\backslash G) \otimes V_{\tau_n})^K \oplus (L^2_c(\Gamma\backslash G) \otimes V_{\tau_n})^K. \tag{15}$$

The continuous part $(L^2_c(\Gamma\backslash G) \otimes V_{\tau_n})^K$ is spanned by the wave packets with the Eisenstein series, so we need to know how D acts on the Eisenstein series. For $\Phi \in (\mathcal{H}_\Gamma(\sigma_+, \nu) \otimes V_{\tau_n})^K$, the Eisenstein series attached to Φ is defined by

$$E(\Phi : \nu : x) := \sum_{\gamma \in \Gamma \cap P \backslash \Gamma} \Phi(\gamma x),$$

which is defined *a priori* for $\mathrm{Re}(\nu) \gg 0$ and has the meromorphic extension over $\mathbb{C}$. Assume that $\Phi_{j,\pm}$ is in the $\pm i$-eigenspace of $c(H)$ on $(\mathcal{H}_\Gamma(\sigma_+, \nu) \otimes V_{\tau_n})^K$ for $j = 1, \ldots, d(\sigma_+)$. The Eisenstein series $E(\Phi_{j,\pm} : i\lambda : x)$ for $\nu = i\lambda$ satisfies

$$DE(\Phi_{j,\pm} : i\lambda : x) = \pm\lambda E(\Phi_{j,\pm} : i\lambda : x). \tag{16}$$

For $\phi \in C_0^\infty(X, E)$, the decomposition (15) provides us with the formula

$$\begin{aligned} \phi(x) &= \sum_k (\phi, \phi_k)\phi_k(x) \\ &+ \frac{1}{4\pi} \sum_{j=1}^{d(\sigma_+)} \int_{-\infty}^{\infty} E(\Phi_{j,+} : i\lambda : x) \int_X E(\Phi_{j,+} : -i\lambda : y)\phi(y)\, dy\, d\lambda \\ &+ \frac{1}{4\pi} \sum_{j=1}^{d(\sigma_+)} \int_{-\infty}^{\infty} E(\Phi_{j,-} : i\lambda : x) \int_X E(\Phi_{j,-} : -i\lambda : y)\phi(y)\, dy\, d\lambda, \end{aligned} \tag{17}$$

where $\{\phi_k\}$ is an orthonormal basis of $(L^2_d(\Gamma\backslash G) \otimes V_{\tau_n})^K$. Since $\mathcal{D}$ preserves the decomposition of (15), we may assume that each ϕ_k is an eigensection of $\mathcal{D}$. Therefore (16) and (17) imply

$$\begin{aligned} \mathcal{D}\phi(x) &= \sum_{\lambda_k} \lambda_k(\phi, \phi_k)\phi_k(x) \\ &+ \frac{1}{4\pi} \sum_{j=1}^{d(\sigma_+)} \int_{-\infty}^{\infty} \lambda E(\Phi_{j,+} : i\lambda : x) \int_X E(\Phi_{j,+} : -i\lambda : y)\phi(y)\, dy\, d\lambda \\ &- \frac{1}{4\pi} \sum_{j=1}^{d(\sigma_+)} \int_{-\infty}^{\infty} \lambda E(\Phi_{j,-} : i\lambda : x) \int_X E(\Phi_{j,-} : -i\lambda : y)\phi(y)\, dy\, d\lambda. \end{aligned}$$

From this we can see that the following equalities hold in the distributional sense:

$$(18)\quad e^{-t\mathcal{D}^2}(x:y) = \sum_{\lambda_k\in\sigma_p} e^{-t\lambda_k^2}\phi_k(x)\otimes\phi_k^*(y)$$
$$+\frac{1}{4\pi}\sum_{j=1}^{d(\sigma_+)}\int_{-\infty}^{\infty} e^{-t\lambda^2}E(\Phi_{j,+}:i\lambda:x)\otimes E(\Phi_{j,+}:-i\lambda:y)^*\,d\lambda$$
$$+\frac{1}{4\pi}\sum_{j=1}^{d(\sigma_+)}\int_{-\infty}^{\infty} e^{-t\lambda^2}E(\Phi_{j,-}:i\lambda:x)\otimes E(\Phi_{j,-}:-i\lambda:y)^*\,d\lambda,$$

$$(19)\quad \mathcal{D}e^{-t\mathcal{D}^2}(x:y) = \sum_{\lambda_k\in\sigma_p} \lambda_k e^{-t\lambda_k^2}\phi_k(x)\otimes\phi_k^*(y)$$
$$+\frac{1}{4\pi}\sum_{j=1}^{d(\sigma_+)}\int_{-\infty}^{\infty} \lambda e^{-t\lambda^2}E(\Phi_{j,+}:i\lambda:x)\otimes E(\Phi_{j,+}:-i\lambda:y)^*d\lambda$$
$$-\frac{1}{4\pi}\sum_{j=1}^{d(\sigma_+)}\int_{-\infty}^{\infty} \lambda e^{-t\lambda^2}E(\Phi_{j,-}:i\lambda:x)\otimes E(\Phi_{j,-}:-i\lambda:y)^*d\lambda.$$

Recalling the decomposition (1) of X, we consider the subset $W_{i,R} := [R,\infty)\times N_i$ in W_i. We choose R_0 such that $W_{i,R}$'s are disjoint each other for $R \geq R_0$. We put $W_R := \cup_{i=1}^{\kappa} W_{i,R}$ and $X_R := X - W_R$ for $R \geq R_0$. The constant term of the Eisenstein series $E(\Phi : i\lambda : x)$ over W_i has the form

$$(20)\qquad e^{(-i\lambda+n)r}\Phi_i + e^{(i\lambda+n)r}\big((C_\Gamma(\sigma_+, i\lambda)\otimes \mathrm{Id})\Phi\big)_i.$$

Here Φ_i is the component of Φ over W_i and note that the operator $C_\Gamma(\sigma_+, i\lambda)\otimes\mathrm{Id}$ acts on $(\mathcal{H}_\Gamma(\sigma_+, i\lambda)\otimes V_{\tau_n})^K$. From now on, we assume that $||\Phi_i|| = 1$ for $i = 1,\ldots,\kappa$. We now discuss the *Maass-Selberg relation* in our context.

PROPOSITION 5.1. (Maass-Selberg) *We have*

$$(21)\quad \int_{X_R} |E(\Phi_{j,\pm}:i\lambda:x)|^2 dx = 2\kappa R - \mathrm{tr}\big(C_\pm(i\lambda)C'_\mp(-i\lambda)\big) + O(e^{-cR})$$
$$= 2\kappa R - \mathrm{tr}\big(C_\mp(-i\lambda)C'_\pm(i\lambda)\big) + O(e^{-cR})$$

where c is a positive constant.

Proof. We will consider only the case of Φ_+ so we use the notation Φ instead of Φ_+ in the following proof. The case of Φ_- can be done in the same way. It follows from Green's formula that

$$(22)\quad \begin{aligned}(\lambda-\lambda')\langle E(\Phi:i\lambda:x),E(\Phi:i\lambda':x)\rangle_{X_R}\\ &= \langle DE(\Phi:i\lambda:x),E(\Phi:i\lambda':x)\rangle_{X_R}-\langle E(\Phi:i\lambda:x),DE(\Phi:i\lambda':x)\rangle_{X_R}\\ &= \langle c(H)E(\Phi:i\lambda:x),E(\Phi:i\lambda':x)\rangle_{\partial X_R}.\end{aligned}$$

By (20), we have

$$\begin{aligned}&\langle c(H)E(\Phi:i\lambda:x),E(\Phi:i\lambda':x)\rangle_{\partial W_{i,R}}\\ &= i\langle e^{(-i\lambda+n)R}\Phi_i, e^{(-i\lambda'+n)R}\Phi_i\rangle_{\partial W_{i,R}}\\ &\quad -i\langle e^{(i\lambda+n)R}((C_\Gamma(\sigma_+,i\lambda)\otimes\mathrm{Id})\Phi)_i, e^{(i\lambda'+n)R}((C_\Gamma(\sigma_+,i\lambda')\otimes\mathrm{Id})\Phi)_i\rangle_{\partial W_{i,R}}+O(e^{-cR})\\ &= ie^{-i(\lambda-\lambda')R}-ie^{i(\lambda-\lambda')R}\sum_{k=1}^{\kappa}C_+(i\lambda)_{ik}C_-(-i\lambda')_{ki}+O(e^{-cR})\end{aligned}$$

where $C_\pm(i\lambda)_{ik}$ is a component of $C_\pm(i\lambda)$. Now we use the functional equation

$$C_\Gamma(\sigma_+,i\lambda)C_\Gamma(\sigma_+,-i\lambda)=\begin{pmatrix}0 & C_-(i\lambda)\\ C_+(i\lambda) & 0\end{pmatrix}\begin{pmatrix}0 & C_-(-i\lambda)\\ C_+(-i\lambda) & 0\end{pmatrix}=\mathrm{Id},$$

which implies the equality

$$(23)\quad \begin{aligned}\langle c(H)E(\Phi:i\lambda:x),E(\Phi:i\lambda':x)\rangle_{\partial W_{i,R}}\\ &= ie^{-i(\lambda-\lambda')R}-i\,e^{i(\lambda-\lambda')R}\\ &\quad -ie^{i(\lambda-\lambda')R}\sum_{k=1}^{\kappa}C_+(i\lambda)_{ik}C_-(-i\lambda')_{ki}\\ &\quad +ie^{i(\lambda-\lambda')R}\sum_{k=1}^{\kappa}C_+(i\lambda)_{ik}C_-(-i\lambda)_{ki}+O(e^{-cR}).\end{aligned}$$

Combining (22) and (23), we have

$$(24)\quad \begin{aligned}(\lambda-\lambda')\langle E(\Phi:i\lambda:x),E(\Phi:i\lambda':x)\rangle_{X_R} &= i\kappa e^{-i(\lambda-\lambda')R}-i\kappa e^{i(\lambda-\lambda')R}\\ &\quad + i\,e^{i(\lambda-\lambda')R}\sum_{i,k=1}^{\kappa}C_+(i\lambda)_{ik}\big(C_-(-i\lambda)_{ki}-C_-(-i\lambda')_{ki}\big)+O(e^{-cR}).\end{aligned}$$

If we pass to the limit $\lambda\to\lambda'$ after dividing each side of (24) by $\lambda-\lambda'$, we get the equality (21). □

The next task is to consider the corresponding formula for $\mathcal{D}_0(i)$. We fix a Lagrangian subspace L in V_{τ_n} as before. For an orthonormal basis $\{\psi_j\} \subset L$, we define

$$\phi_{j,+} := \frac{1}{\sqrt{2}}\Big(\psi_j - ic(H)\psi_j\Big), \quad \phi_{j,-} := \frac{1}{\sqrt{2}}\Big(\psi_j + ic(H)\psi_j\Big)$$

and

$$\begin{aligned} e(\phi_{j,+} : i\lambda : x) &:= e^{(n-i\lambda)r}\phi_{j,+} + e^{(n+i\lambda)r}\phi_{j,-}, \\ e(\phi_{j,-} : i\lambda : x) &:= e^{(n-i\lambda)r}\phi_{j,-} + e^{(n+i\lambda)r}\phi_{j,+}. \end{aligned}$$

Note that $e(\phi_{j,\pm} : i\lambda : x)$ lies in $C^\infty(\mathbb{R}^+, V_{\tau_n}, L) := \{\ \phi \in C^\infty(\mathbb{R}^+, E_0) \mid \phi(0) \in L\ \}$ and

$$\begin{aligned} c(H)\left(\frac{d}{dr} - n\,\mathrm{Id}\right) e(\phi_{j,+} : i\lambda : x) &= \lambda\, e(\phi_{j,+} : i\lambda : x), \\ c(H)\left(\frac{d}{dr} - n\,\mathrm{Id}\right) e(\phi_{j,-} : i\lambda : x) &= -\lambda\, e(\phi_{j,-} : i\lambda : x). \end{aligned}$$

As when we introduced $\mathcal{D}_0(i)$, we can regard $e(\phi_{j,\pm} : i\lambda : x)$ as lying in $W_i \subset X$ and denote such a section by $E^i(\phi_{j,\pm} : i\lambda : x)$. Then, for $\phi = (\phi_0, \phi_c) \in L^2(X - W_i, E) \oplus L^2(W_i, E)$ with $\phi_c \in C_0^\infty(W_i, E)$ and $\phi_c|_{\partial W_i} \in L$,

$$\begin{aligned} (25)\quad \mathcal{D}_0(i)\phi = &\ \frac{1}{4\pi}\sum_{j=1}^{d(\sigma_+)}\int_{-\infty}^{\infty} \lambda E^i(\phi_{j,+} : i\lambda : x)\int_X E^i(\phi_{j,+} : -i\lambda : y)\phi(y)dy\, d\lambda \\ &-\frac{1}{4\pi}\sum_{j=1}^{d(\sigma_+)}\int_{-\infty}^{\infty} \lambda E^i(\phi_{j,-} : i\lambda : x)\int_X E^i(\phi_{j,-} : -i\lambda : y)\phi(y)dy\, d\lambda. \end{aligned}$$

The following equality can be proved in the same way as in the proof of Proposition 5.1,

$$(26)\qquad \sum_{i=1}^{\kappa}\int_{X_R} |E^i(\phi_{j,\pm} : i\lambda : x)|^2 dx = 2\kappa R + O(e^{-cR})$$

for some positive constant c. (Let us remark that there is no contribution over ∂W_i by the choice of $\phi_{j,\pm}$ when we apply the Green formula as in (22).) It follows from (21), (26) that

$$\begin{aligned} (27)\quad &\int_X |E(\Phi_{j,\pm} : i\lambda : x)|^2 - \sum_{i=1}^{\kappa} |E^i(\phi_{j,\pm} : i\lambda : x)|^2 dx \\ &= \lim_{R\to\infty}\int_{X_R} |E(\Phi_{j,\pm} : i\lambda : x)|^2 - \sum_{i=1}^{\kappa} |E^i(\phi_{j,\pm} : i\lambda : x)|^2 dx \\ &= -\mathrm{tr}(C_{\mp}(-i\lambda)C'_{\pm}(i\lambda)). \end{aligned}$$

Finally, the following proposition is the result of (5), (6), (8), (18), (19), (25), (27) and Corollary 4.3.

PROPOSITION 5.2. *We have*

$$\text{(28)}\qquad \operatorname{Tr}\left(e^{-t\mathcal{D}^2}-\sum_{i=1}^{\kappa}e^{-t\mathcal{D}_0^2(i)}\right)$$
$$=\sum_{\lambda_k}e^{-t\lambda_k^2}-\frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty}e^{-t\lambda^2}\operatorname{tr}(C_\Gamma(\sigma_+,-i\lambda)C'_\Gamma(\sigma_+,i\lambda))\,d\lambda$$
$$=I_\Gamma(K_t^e)+H_\Gamma(K_t^e)+U_\Gamma(K_t^e),$$

$$\text{(29)}\quad \operatorname{Tr}\left(\mathcal{D}e^{-t\mathcal{D}^2}-\sum_{i=1}^{\kappa}\mathcal{D}_0(i)e^{-t\mathcal{D}_0^2(i)}\right)$$
$$=\sum_{\lambda_k}\lambda_k e^{-t\lambda_k^2}-\frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty}\lambda e^{-t\lambda^2}\operatorname{tr}_s(C_\Gamma(\sigma_+,-i\lambda)C'_\Gamma(\sigma_+,i\lambda))\,d\lambda$$
$$=\sum_{\lambda_k}\lambda_k e^{-t\lambda_k^2}-\frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty}\lambda e^{-t\lambda^2}\operatorname{tr}(C_-(-i\lambda)C'_+(i\lambda))\,d\lambda$$
$$+\frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty}\lambda e^{-t\lambda^2}\operatorname{tr}(C_+(-i\lambda)C'_-(i\lambda))\,d\lambda$$
$$=H_\Gamma(K_t^o).$$

Remark 5.3. Although the definition of $\mathcal{D}_0(i)$ depends on the choice of the Lagrangian subspace L, the relative traces on the left sides of (28), (29) do not depend on this choice. This is because the right side of (26) does not depend on the choice of L.

6. Meromorphic continuations of the eta and zeta functions. In this section, we prove Theorem 1.1, which provides us with the pole structures of the eta function and the zeta function over $\mathbb{C}$. We follow [18] and [19] and define

$$\text{(30)}\qquad \eta_{\mathcal{D},1}(z):=\frac{1}{\Gamma(\frac{z+1}{2})}\int_0^1 t^{\frac{z-1}{2}}\operatorname{Tr}\left(\mathcal{D}e^{-t\mathcal{D}^2}-\sum_{i=1}^{\kappa}\mathcal{D}_0(i)e^{-t\mathcal{D}_0^2(i)}\right)dt,$$
$$\zeta_{\mathcal{D}^2,1}(z):=\frac{1}{\Gamma(z)}\int_0^1 t^{z-1}\left[\operatorname{Tr}\left(e^{-t\mathcal{D}^2}-\sum_{i=1}^{\kappa}e^{-t\mathcal{D}_0^2(i)}\right)-h\right]dt$$

for $\mathrm{Re}(z) \gg 0$ and

$$\text{(31)} \quad \eta_{\mathcal{D},2}(z) := \frac{1}{\Gamma(\frac{z+1}{2})} \int_1^\infty t^{\frac{z-1}{2}} \operatorname{Tr}\left(\mathcal{D}e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa} \mathcal{D}_0(i) e^{-t\mathcal{D}_0^2(i)}\right) dt,$$

$$\zeta_{\mathcal{D}^2,2}(z) := \frac{1}{\Gamma(z)} \int_1^\infty t^{z-1} \left[\operatorname{Tr}\left(e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa} e^{-t\mathcal{D}_0^2(i)}\right) - h\right] dt$$

for $\mathrm{Re}(z) \ll 0$ where h is the multiplicity of zero eigenvalues of $\mathcal{D}^2$. To define the eta invariant and the regularized determinant, we need to study the meromorphic extensions of $\eta_{\mathcal{D},i}(z)$ and $\zeta_{\mathcal{D}^2,i}(z)$ near $z = 0$ for $i = 1, 2$. The difficulty, which is not present for closed manifolds, is the presence of continuous spectrum. Moreover, the continuous spectrum of $\mathcal{D}$ is equal to the whole real line in our case. Hence the meromorphic extensions of $\eta_{\mathcal{D},2}(z)$ and $\zeta_{\mathcal{D}^2,2}(z)$ have nontrivial poles.

We start with $\eta_{\mathcal{D},1}(z)$. It follows from (10) and (29) that

$$\text{(32)} \quad \operatorname{Tr}(\mathcal{D}e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa} \mathcal{D}_0(i) e^{-t\mathcal{D}_0^2(i)}) = H_\Gamma(K_t^o)$$
$$= \frac{2\pi i}{(4\pi t)^{\frac{3}{2}}} \sum_{\gamma:\ \text{hyperbolic}} l(C_\gamma)^2 j(\gamma)^{-1} D(\gamma)^{-1} \left(\overline{\chi_{\sigma_+}(m_\gamma)} - \overline{\chi_{\sigma_-}(m_\gamma)}\right) e^{-\frac{l(C_\gamma)^2}{4t}}.$$

The number $c := \min_{\{\gamma:\ \text{hyperbolic}\}} l(C_\gamma)$ is a positive real number, hence as $t \to 0$,

$$\text{(33)} \quad \frac{2\pi i}{(4\pi t)^{\frac{3}{2}}} \sum_{\gamma:\ \text{hyperbolic}} l(C_\gamma)^2 j(\gamma)^{-1} D(\gamma)^{-1} (\overline{\chi_{\sigma_+}(m_\gamma)} - \overline{\chi_{\sigma_-}(m_\gamma)}) e^{-\frac{l(C_\gamma)^2}{4t}} \sim a\, e^{-\frac{c^2}{4t}}$$

for a constant a. Now (32) and (33) give

$$\operatorname{Tr}\left(\mathcal{D}e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa} \mathcal{D}_0(i) e^{-t\mathcal{D}_0^2(i)}\right) \sim a\, e^{-\frac{c^2}{4t}} \quad \text{as} \quad t \to 0.$$

This means that $\eta_{\mathcal{D},1}(z)$ can be extended to the whole complex plane without poles.

For $\zeta_{\mathcal{D}^2,1}(z)$, we use (28) to get

$$\operatorname{Tr}\left(e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa} e^{-t\mathcal{D}_0^2(i)}\right) = I_\Gamma(K_t^e) + H_\Gamma(K_t^e) + U_\Gamma(K_t^e).$$

Recall that

$$\text{(34)} \quad H_\Gamma(K_t^e) = \frac{1}{\sqrt{4\pi t}} \sum_{\gamma:\ \text{hyperbolic}} l(C_\gamma) j(\gamma)^{-1} D(\gamma)^{-1} \left(\overline{\chi_{\sigma_+}(m_\gamma)} + \overline{\chi_{\sigma_-}(m_\gamma)}\right) e^{-\frac{l(C_\gamma)^2}{4t}}.$$

As $H_\Gamma(K_t^o)$, we have

$$H_\Gamma(K_t^e) \sim a' e^{-\frac{c^2}{4t}} \quad \text{as } t \to 0 \tag{35}$$

for a constant a'. We can see that $H_\Gamma(K_t^e)$ does not give any poles in the meromorphic continuation of $\zeta_{\mathcal{D}^2,1}(z)$. Next we consider $I_\Gamma(K_t^e)$ and $U_\Gamma(K_t^e)$. An elementary computation shows

$$I_\Gamma(K_t^e) = \sum_{k=0}^{n} a_k t^{-k-\frac{1}{2}} \tag{36}$$

for some constants a_k. By Corollary 4.3, we have

$$U_\Gamma(K_t^e) = \frac{2}{2\pi}\int_{-\infty}^{\infty} e^{-t\lambda^2}(P_U(\lambda) + Q(\lambda))\, d\lambda$$

where $P_U(\lambda)$ is an even polynomial of degree $(2n-4)$ and $Q(\lambda)$ is given by

$$-\kappa\frac{d(\sigma_+)}{2}\left(\psi\left(i\lambda + \frac{1}{2}\right) + \psi\left(-i\lambda + \frac{1}{2}\right)\right).$$

An elementary computation leads to

$$\int_{-\infty}^{\infty} e^{-t\lambda^2} P_U(\lambda)\, d\lambda = \sum_{k=0}^{n-2} b_k t^{-k-\frac{1}{2}} \tag{37}$$

for some constants b_k. Using the relations

$$\psi(x+1) = \frac{1}{x} + \psi(x), \qquad \psi(x) + \psi\left(x + \frac{1}{2}\right) = 2(\psi(2x) - \log 2),$$

we have

$$Q(\lambda) = -\frac{\kappa d(\sigma_+)}{2}\Big(2\big(\psi(2i\lambda+1)+\psi(-2i\lambda+1)-2\log 2\big) - \big(\psi(i\lambda+1)+\psi(-i\lambda+1)\big)\Big).$$

To deal with the digamma function ψ, we use the following asymptotic expansion

$$\psi(z+1) \sim \log z + \frac{1}{2z} - \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)z^{2k}} \qquad \text{as} \quad z \to \infty$$

where B_{2k} are Bernoulli numbers. This implies the expansion

$$\int_{-\infty}^{\infty} e^{-t\lambda^2} Q(\lambda)\, d\lambda \sim \sum_{k=0}^{\infty} c_k t^{k-\frac{1}{2}} + d_0 t^{-\frac{1}{2}} \log t \qquad \text{as} \quad t \to 0 \tag{38}$$

for some constants c_k and d_0. By (35), (36), (37), (38), we have

$$I_\Gamma(K_t^e) + H_\Gamma(K_t^e) + U_\Gamma(K_t^e) \sim \sum_{k=-n}^{\infty} \beta_k t^{k-\frac{1}{2}} + \beta_0' t^{-\frac{1}{2}} \log t \qquad \text{as} \quad t \to 0$$

for constants β_k, β_0'. Therefore $\zeta_{\mathcal{D}^2,1}(z)$ is well defined for $\mathrm{Re}(z) > n + \frac{1}{2}$ and we can extend $\zeta_{\mathcal{D}^2,1}(z)$ to a meromorphic function on $\mathbb{C}$, with poles determined by

$$\Gamma(z)\zeta_{\mathcal{D}^2,1}(z) = \sum_{k=-n}^{\infty} \frac{\beta_k}{z+k-\frac{1}{2}} + \frac{\beta_0'}{(z-\frac{1}{2})^2} - \frac{h}{z} + H_1(z) \tag{39}$$

where β_k, β_0' are constants and $H_1(z)$ is a holomorphic function.

To deal with the meromorphic extensions of $\eta_{\mathcal{D},2}(z)$ and $\zeta_{\mathcal{D}^2,2}(z)$, we consider the right sides of the following equalities,

$$\begin{aligned}
\mathrm{Tr}\left(\mathcal{D}e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa} \mathcal{D}_0(i) e^{-t\mathcal{D}_0^2(i)}\right) &= \sum_{\lambda_k} \lambda_k e^{-t\lambda_k} \\
&\quad - \frac{d(\sigma_+)}{4\pi} \int_{-\infty}^{\infty} \lambda e^{-t\lambda^2} \mathrm{tr}_s(C_\Gamma(\sigma_+, -i\lambda) C_\Gamma'(\sigma_+, i\lambda))\, d\lambda, \\
\mathrm{Tr}\left(e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa} e^{-t\mathcal{D}_0^2(i)}\right) - h &= \sum_{\lambda_k \neq 0} e^{-t\lambda_k^2} \\
&\quad - \frac{d(\sigma_+)}{4\pi} \int_{-\infty}^{\infty} e^{-t\lambda^2} \mathrm{tr}(C_\Gamma(\sigma_+, -i\lambda) C_\Gamma'(\sigma_+, i\lambda))\, d\lambda.
\end{aligned}$$

The discrete eigenvalues give

$$\sum_{\lambda_k} \lambda_k e^{-t\lambda_k^2} \sim e^{-ct}, \quad \sum_{\lambda_k \neq 0} e^{-t\lambda_k^2} \sim e^{-ct} \qquad \text{as} \quad t \to \infty$$

for a positive constant c. The operator $C_\Gamma(\sigma_+, i\lambda)$ is analytic along the imaginary axis and

$$\begin{aligned}
\mathrm{tr}_s(C_\Gamma(\sigma_+, -i\lambda) C_\Gamma'(\sigma_+, i\lambda)) &= -\mathrm{tr}_s(C_\Gamma(\sigma_+, i\lambda) C_\Gamma'(\sigma_+, -i\lambda)), \\
\mathrm{tr}(C_\Gamma(\sigma_+, -i\lambda) C_\Gamma'(\sigma_+, i\lambda)) &= \mathrm{tr}(C_\Gamma(\sigma_+, i\lambda) C_\Gamma'(\sigma_+, -i\lambda)),
\end{aligned}$$

hence we have the following analytic expansion at $\lambda = 0$:

$$\begin{aligned}
\mathrm{tr}_s(C_\Gamma(\sigma_+, -i\lambda) C_\Gamma'(\sigma_+, i\lambda)) &= \sum_{k=0}^{\infty} f_{2k+1} \lambda^{2k+1}, \\
\mathrm{tr}(C_\Gamma(\sigma_+, -i\lambda) C_\Gamma'(\sigma_+, i\lambda)) &= \sum_{k=0}^{\infty} g_{2k} \lambda^{2k}
\end{aligned}$$

for some constants f_{2k+1}, g_{2k}. Therefore we have

$$(40)\quad \int_{-1}^{1} \lambda e^{-t\lambda^2} \mathrm{tr}_s(C_\Gamma(\sigma_+,-i\lambda)C'_\Gamma(\sigma_+,i\lambda))d\lambda \sim \sum_{k=0}^{\infty} \gamma_k t^{-(k+\frac{3}{2})} \quad \text{as } t \to \infty,$$

$$(41)\quad \int_{-1}^{1} e^{-t\lambda^2} \mathrm{tr}(C_\Gamma(\sigma_+,-i\lambda)C'_\Gamma(\sigma_+,i\lambda))\, d\lambda \sim \sum_{k=0}^{\infty} \gamma'_k t^{-(k+\frac{1}{2})} \quad \text{as } t \to \infty.$$

The corresponding integrals over $(-\infty, 1]_\lambda \cup [1, \infty)_\lambda$ converge to 0 exponentially as $t \to \infty$. Hence, the expansion (40) shows that $\eta_{\mathcal{D},2}(z)$ is well defined for $\mathrm{Re}(z) < 1$ and extends to the whole complex plane with the following pole structure:

$$(42)\qquad \Gamma((z+1)/2)\eta_{\mathcal{D},2}(z) = \sum_{k=0}^{\infty} \frac{-2\gamma_k}{z-2k-2} + K_2(z)$$

for constants γ_k and a holomorphic function $K_2(z)$. In the same way, the expansion (41) implies that $\zeta_{\mathcal{D}^2,2}(z)$ is well defined for $\mathrm{Re}(z) < \frac{1}{2}$ and extends to the whole complex plane with poles determined by the equality

$$(43)\qquad \Gamma(z)\zeta_{\mathcal{D}^2,2}(z) = \sum_{k=0}^{\infty} \frac{-\gamma'_k}{z-k-\frac{1}{2}} + H_2(z)$$

for the constants γ'_k and a holomorphic function $H_2(z)$.

We define the eta and zeta functions by

$$(44)\qquad \eta_{\mathcal{D}}(z) := \eta_{\mathcal{D},1}(z) + \eta_{\mathcal{D},2}(z), \quad \zeta_{\mathcal{D}^2}(z) := \zeta_{\mathcal{D}^2,1}(z) + \zeta_{\mathcal{D}^2,2}(z).$$

Here now the right sides of these equalities are meromorphic functions over $\mathbb{C}$ with the poles described in the above. The equalities (39), (42), (43) give the following theorem:

THEOREM 6.1. *The poles of the eta function $\eta_{\mathcal{D}}(z)$ and the zeta function $\zeta_{\mathcal{D}^2}(z)$ are determined by the equations*

$$\Gamma((z+1)/2)\eta_{\mathcal{D}}(z) = \sum_{k=0}^{\infty} \frac{-2\gamma_k}{z-2k-2} + K(z),$$

$$\Gamma(z)\zeta_{\mathcal{D}^2}(z) = \sum_{k=-n}^{\infty} \frac{\beta_k}{z+k-\frac{1}{2}} + \frac{\beta'_0}{(z-\frac{1}{2})^2} - \frac{h}{z} + \sum_{k=0}^{\infty} \frac{-\gamma'_k}{z-k-\frac{1}{2}} + H(z)$$

where $K(z)$ and $H(z)$ are holomorphic. In particular, $\eta_{\mathcal{D}}(z)$ and $\zeta_{\mathcal{D}^2}(z)$ are regular at $z = 0$.

7. Eta invariants, Zeta functions of odd type and functional equations. In this section, we study the eta invariant and its relation with the Selberg zeta function of odd type. We use the Selberg trace formula to prove a generalization of Millson's theorem in [12] for hyperbolic manifolds with cusps. Since the unipotent term in our situation vanishes (see Corollary 4.3), we obtain the same formula as in the case of Millson in [12]. We also derive the functional equation for the eta invariant and the Selberg zeta function of odd type.

By the analysis in Section 6, the eta function $\eta_{\mathcal{D}}(z)$ is regular at $z = 0$ and we can put $z = 0$ in the equality

$$\eta_{\mathcal{D}}(z) = \frac{1}{\Gamma(\frac{z+1}{2})} \int_0^{\infty} t^{\frac{z-1}{2}} \mathrm{Tr}\left(\mathcal{D}e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa} \mathcal{D}_0(i) e^{-t\mathcal{D}_0^2(i)} \right) dt.$$

We define the eta invariant of $\mathcal{D}$ by

$$\eta(\mathcal{D}) := \eta_{\mathcal{D}}(0) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} t^{-\frac{1}{2}} \mathrm{Tr}\left(\mathcal{D}e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa} \mathcal{D}_0(i) e^{-t\mathcal{D}_0^2(i)} \right) dt.$$

Let us recall that

$$\begin{aligned} &\mathrm{Tr}\left(\mathcal{D}e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa} \mathcal{D}_0(i) e^{-t\mathcal{D}_0^2(i)} \right) \\ &\quad = \frac{2\pi i}{(4\pi t)^{\frac{3}{2}}} \sum_{\gamma:\ \text{hyperbolic}} l(C_\gamma)^2 j(\gamma)^{-1} D(\gamma)^{-1} \left(\overline{\chi_{\sigma_+}(m_\gamma)} - \overline{\chi_{\sigma_-}(m_\gamma)}\right) e^{-\frac{l(C_\gamma)^2}{4t}}. \end{aligned}$$

Using the elementary equality $\int_0^{\infty} e^{-s^2 t} \frac{e^{-r^2/4t}}{(4\pi t)^{\frac{3}{2}}} dt = \frac{e^{-sr}}{4\pi r}$, we have

$$\begin{aligned} &\int_0^{\infty} e^{-s^2 t} \mathrm{Tr}\left(\mathcal{D}e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa} \mathcal{D}_0(i) e^{-t\mathcal{D}_0^2(i)} \right) dt \\ &\quad = \frac{i}{2} \sum_{\gamma:\ \text{hyperbolic}} l(C_\gamma) j(\gamma)^{-1} D(\gamma)^{-1} \left(\overline{\chi_{\sigma_+}(m_\gamma)} - \overline{\chi_{\sigma_-}(m_\gamma)}\right) e^{-sl(C_\gamma)} \\ &\quad = \frac{i}{2} \sum_{\gamma:\ \text{hyperbolic}} l(C_\gamma) j(\gamma)^{-1} |\det(\mathrm{Ad}(a_\gamma m_\gamma)^{-1} - I|_{\mathfrak{n}})|^{-1} \\ &\qquad\qquad \times \left(\overline{\chi_{\sigma_+}(m_\gamma)} - \overline{\chi_{\sigma_-}(m_\gamma)}\right) e^{-(s+n)l(C_\gamma)}. \end{aligned}$$

We define the Selberg zeta function of odd type by

$$Z_H^o(s) := \exp\left(-\sum_{\gamma:\ \text{hyperbolic}} j(\gamma)^{-1}|\det(\mathrm{Ad}(a_\gamma m_\gamma)^{-1} - I|_{\mathfrak{n}})|^{-1} \times (\overline{\chi_{\sigma_+}(m_\gamma)} - \overline{\chi_{\sigma_-}(m_\gamma)})e^{-sl(C_\gamma)}\right) \tag{45}$$

for $\mathrm{Re}(s) \gg 0$. In Proposition 7.2 we will show that $Z_H^o(s)$ has a meromorphic extension over $\mathbb{C}$ and $Z_H^o(s)$ is regular at $s = n$. Now we have

$$\int_0^\infty e^{-s^2t}\mathrm{Tr}(\mathcal{D}e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa} \mathcal{D}_0(i)e^{-t\mathcal{D}_0^2(i)})\, dt = \frac{i}{2}\frac{d}{ds}\log Z_H^o(s+n).$$

Following the argument on p. 27 of [12], we use the equality

$$t^{\frac{z-1}{2}} = \frac{2}{\Gamma(\frac{1-z}{2})}\int_0^\infty s^{-z}e^{-s^2t}\, ds$$

for $\mathrm{Re}(z) < 1$ to get

$$\begin{aligned}
\eta_{\mathcal{D}}(z) &= \frac{2}{\Gamma(\frac{z+1}{2})\Gamma(\frac{1-z}{2})}\int_0^\infty \int_0^\infty s^{-z}e^{-s^2t}\, ds\, \mathrm{Tr}\left(\mathcal{D}e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa}\mathcal{D}_0(i)e^{-t\mathcal{D}_0^2(i)}\right) dt \\
&= \frac{2}{\Gamma(\frac{z+1}{2})\Gamma(\frac{1-z}{2})}\int_0^\infty s^{-z}\int_0^\infty e^{-s^2t}\mathrm{Tr}\left(\mathcal{D}e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa}\mathcal{D}_0(i)e^{-t\mathcal{D}_0^2(i)}\right) dt\, ds \\
&= \frac{i}{\Gamma(\frac{z+1}{2})\Gamma(\frac{1-z}{2})}\int_0^\infty s^{-z}\frac{d}{ds}\log Z_H^o(s+n)\, ds.
\end{aligned}$$

If we evaluate the above equality at $z = 0$, we get the following theorem.

THEOREM 7.1. *For the Dirac operator $\mathcal{D}$ over a $(2n+1)$-dimensional hyperbolic manifold with cusps, the eta invariant $\eta(\mathcal{D})$ and the Selberg zeta function of odd type $Z_H^o(s)$ satisfy*

$$\eta(\mathcal{D}) = \frac{1}{\pi i}\log Z_H^o(n). \tag{46}$$

Now let us show that $Z_H^o(s)$ has the meromorphic extension over $\mathbb{C}$. We select a smooth odd function $g(u)$ such that $|g(u)| = 1$ if $|u| > c$, $g(u) = 0$ near 0 and

$\int_0^\infty g'(u)du = 1$ where $c = \min_{\{\gamma:\text{ hyperbolic}\}} l(C_\gamma)$. We define

$$\begin{cases} H_s(\sigma_+, \lambda) := \int_{-\infty}^{\infty} g(u)e^{-s|u|}e^{i\lambda u}du, \\ H_s(\sigma_-, \lambda) := -H_s(\sigma_+, \lambda) \end{cases}$$

for a complex parameter s. By the Paley-Wiener theorem, (see theorem 2.2 in [12]), there exists f_s over G with $\hat{f}_s(\sigma_\pm, i\lambda) = H_s(\sigma_\pm, \lambda)$, $\hat{f}_s(\sigma, i\lambda) = 0$ if $\sigma \neq \sigma_\pm$. Applying the Selberg trace formula to the one parameter family of functions f_s on G for $\mathrm{Re}(s) \gg 0$, we get

$$\begin{aligned}
(47)\qquad & \sum_{\lambda_j \in \sigma_p^+} H_s(\sigma_+, \lambda_j) + \sum_{\lambda_j \in \sigma_p^-} H_s(\sigma_-, \lambda_j) \\
& \quad - \frac{d(\sigma_+)}{4\pi} \int_{-\infty}^{\infty} H_s(\sigma_+, \lambda) \mathrm{tr}_s(C_\Gamma(\sigma_+, -i\lambda) C'_\Gamma(\sigma_+, i\lambda)) d\lambda \\
& = \sum_{\gamma:\text{ hyperbolic}} l(C_\gamma) j(\gamma)^{-1} D(\gamma)^{-1} \left(\overline{\chi_{\sigma_+}(m_\gamma)} - \overline{\chi_{\sigma_-}(m_\gamma)} \right) e^{-sl(C_\gamma)} \\
& = \frac{d}{ds} \log Z_H^o(s+n).
\end{aligned}$$

Here we used the fact that the identity, unipotent orbital integrals vanish by the definition of $H_s(\sigma_\pm, \lambda)$ and results in the previous sections. We shall use this equality to get the meromorphic extension of

$$\mathcal{Z}(s) := \frac{d}{ds} \log Z_H^o(s+n)$$

over $\mathbb{C}$ and to investigate its poles.

Discrete eigenvalue term. Integration by part gives

$$H_s(\sigma_\pm, \lambda) = \pm \frac{1}{s - i\lambda} \int_0^\infty g'(u) e^{-(s-i\lambda)u} du \mp \frac{1}{s + i\lambda} \int_0^\infty g'(u) e^{-(s+i\lambda)u} du.$$

This equality provides the meromorphic extension of $H_s(\sigma_\pm, \lambda_j)$ over $\mathbb{C}$ and we see that $H_s(\sigma_\pm, \lambda_j)$ has simple poles at $i\lambda_j$, $-i\lambda_j$ with residues $\pm m_j$, $\mp m_j$ for $\lambda_j \in \sigma_p^\pm$ where m_j is the multiplicity of λ_j.

Scattering term. We consider the scattering term

$$(48)\qquad - \frac{d(\sigma_+)}{4\pi} \int_{-\infty}^{\infty} H_s(\sigma_+, \lambda) \mathrm{tr}_s(C_\Gamma(\sigma_+, -i\lambda) C'_\Gamma(\sigma_+, i\lambda))\, d\lambda.$$

First we observe

$$\Phi(z) := \operatorname{tr}_s(C_\Gamma(\sigma_+, -z)\frac{d}{dz}C_\Gamma(\sigma_+, z)) = \frac{d}{dz}\left(\log\det C_+(z) - \log\det C_-(z)\right).$$

Using the equalities $C_\Gamma(\sigma_+, z)C_\Gamma(\sigma_+, -z) = \mathrm{Id}$, $C_\Gamma(\sigma_+, z)^* = C_\Gamma(\sigma_+, \bar{z})$ and the first displayed formula on p. 518 and (6.8) of [16], we have

$$\det C_+(z) = \det C_+(0)\, p_+^z \prod_{\mathrm{Re}(q_+)<0} \frac{z+\bar{q}_+}{z-q_+},$$

$$\det C_-(z) = \det C_-(0)\, p_-^z \prod_{\mathrm{Re}(q_-)<0} \frac{z+\bar{q}_-}{z-q_-}$$

for some constants p_+, p_-. Here the infinite products are taken over the poles $\{q_\pm\}$ of $\det C_\pm(z)$ respectively. Note that $\det C_\pm(z)$ is holomorphic over the half plane with $\mathrm{Re}(z) \geq 0$. Hence $\Phi(z)$ has the following form over $\mathbb{C}$:

$$\Phi(z) = -\sum_{\mathrm{Re}(q_+)<0} \frac{2\,\mathrm{Re}(q_+)}{(z-q_+)(z+\bar{q}_+)} + \sum_{\mathrm{Re}(q_-)<0} \frac{2\,\mathrm{Re}(q_-)}{(z-q_-)(z+\bar{q}_-)} + \log p_+ - \log p_-. \tag{49}$$

Now we consider the contour integral

$$\mathcal{L}_R := \frac{1}{4\pi i}\int_{L_R} H_s(\sigma_+, z)\, \operatorname{tr}_s(C_\Gamma(\sigma_+, -iz)C'_\Gamma(\sigma_+, iz))\, dz$$

where $L_R = [\,-R,\ R\,] \cup \{\, Re^{i\theta} \mid 0 \leq \theta \leq \pi \,\}$. As in proposition 3.10 of [8], we can apply the Cauchy integral formula and obtain

$$\lim_{R\to\infty} \mathcal{L}_R = i\left(\sum_{\mathrm{Re}(q_+)<0} \frac{1}{s-q_+}\int_0^\infty g'(u)e^{-(s-q_+)u}du - \sum_{\mathrm{Re}(q_-)<0} \frac{1}{s-q_-}\int_0^\infty g'(u)e^{-(s-q_-)u}du\right) \tag{50}$$

for $\mathrm{Re}(s) \gg 0$. Now the right side of (50) gives us the meromorphic extension over $\mathbb{C}$. On the other hand, we can also show that

$$\lim_{R\to\infty} \mathcal{L}_R = \frac{1}{4\pi i}\int_{-\infty}^{\infty} H_s(\sigma_+, \lambda)\operatorname{tr}_s(C_\Gamma(\sigma_+, -i\lambda)C'_\Gamma(\sigma_+, i\lambda))\, d\lambda.$$

Here the integral over the semicircle of radius R vanishes as $R \to \infty$ by the definition of $H_s(\sigma_+, z)$ and (49). Therefore the meromorphic extension of the scattering term has simple poles at $q_\pm$ for $\mathrm{Re}(q_\pm) < 0$ with residues $\pm d(\sigma_+)b_\pm$ where $b_\pm$ denotes the order of the pole of $\det C_\Gamma(\sigma_+, z)$ at $q_\pm$.

Combining the contributions of the $H_s(\sigma_+, \lambda_j)$'s and the scattering term, we see that $\mathcal{Z}(s)$ has a meromorphic extension over $\mathbb{C}$ and its simple poles are located at $\pm i\lambda_j$, $\mp i\lambda_j$ for $\lambda_j \in \sigma_p^\pm$, and at $q_\pm$ for $\mathrm{Re}(q_\pm) < 0$ with residues m_j, $-m_j$ and $\pm d(\sigma_+)b_\pm$ respectively. In particular, we can see that $\mathcal{Z}(s)$ is regular at $s = 0$. This implies the following proposition.

PROPOSITION 7.2. *The Selberg zeta function of odd type $Z_H^o(s+n)$ has a meromorphic extension over $\mathbb{C}$, and is regular at $s = 0$.*

Remark 7.3. The zeros of $Z_H^o(s+n)$ are located at $\pm i\lambda_j$ for $\lambda_j \in \sigma_p^\pm$, at q_+ for $\mathrm{Re}(q_+) < 0$ and their orders are m_j, $d(\sigma_+)b_+$. The poles of $Z_H^o(s+n)$ are located at $\mp i\lambda_j$ for $\lambda_j \in \sigma_p^\pm$, at q_- for $\mathrm{Re}(q_-) < 0$ and their orders are m_j, $d(\sigma_+)b_-$.

Let us study the functional equation of $\eta(\mathcal{D})$ and $Z_H^o(s)$. We set

$$R(s) := \mathcal{Z}(s) - \mathcal{Z}(-s) + d(\sigma_+)\Phi(s).$$

Then $R(s)$ is an odd entire function of s. Let $h(s)$ be an odd function which decreases sufficiently rapidly as $\mathrm{Im}(s) \to \infty$ in the strip $\{ s \in \mathbb{C} \mid |\mathrm{Re}(s)| < n + \epsilon,\ \epsilon > 0 \}$ and consider the contour integral

$$\mathcal{L}_T := \frac{1}{2\pi i} \int_{L_T} h(s)\mathcal{Z}(s)\, ds$$

where L_T is the rectangle with the corners $a + iT, a - iT, -a + iT, -a - iT$ with $n < a < n + \epsilon$. Then we have

$$\lim_{T\to\infty} \mathcal{L}_T = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} h(s)(\mathcal{Z}(s) - \mathcal{Z}(-s))\, ds + 2\frac{1}{2\pi i} \int_{-a+i\infty}^{-a-i\infty} h(s)\mathcal{Z}(s)\, ds.$$

We apply the Cauchy integral theorem to get the equality

$$\lim_{T\to\infty} \mathcal{L}_T = 2\sum_{\lambda_j} m_j h(i\lambda_j) + \sum_{-a<\mathrm{Re}(q_k)<0} d(\sigma_+)b_k h(q_k), \tag{51}$$

where we use the notations q_k, b_k instead of $q_\pm$, $\pm b_\pm$. Because the simple poles of $\mathcal{Z}(s)$ are located at the q_k's with residues $d(\sigma_+)b_k$, between $\mathrm{Re}(s) = -a$ and $\mathrm{Re}(s) = 0$, we have

$$(52)\qquad \frac{1}{2\pi i}\int_{-a+i\infty}^{-a-i\infty} h(s)\mathcal{Z}(s)\,ds = \frac{1}{2\pi i}\int_{i\infty}^{-i\infty} h(\lambda)\mathcal{Z}(\lambda)d\lambda + \sum_{-a<\mathrm{Re}(q_k)<0} d(\sigma_+)b_k h(q_k).$$

The simple poles at $-q_k$ of $\mathcal{Z}(-s)$ between $\mathrm{Re}(s) = a$ and $\mathrm{Re}(s) = 0$ give rise to the equality

$$(53)\quad \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} h(s)(\mathcal{Z}(s) - \mathcal{Z}(-s))ds = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} h(\lambda)(\mathcal{Z}(\lambda) - \mathcal{Z}(-\lambda))d\lambda + \sum_{0<\mathrm{Re}(-q_k)<a} d(\sigma_+)b_k h(-q_k).$$

By (51), (52) and (53), we have

$$(54)\quad 2\sum_{\lambda_j} m_j h(i\lambda_j) = 2\frac{1}{2\pi i}\int_{i\infty}^{-i\infty} h(\lambda)\Big(\mathcal{Z}(\lambda) + \frac{d(\sigma_+)}{2}\Phi(\lambda)\Big)\,d\lambda + \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} h(\lambda)(\mathcal{Z}(\lambda) - \mathcal{Z}(-\lambda) + d(\sigma_+)\Phi(\lambda))\,d\lambda.$$

If we change variables $\lambda \to i\lambda$, we see that the first term of the right side in (54) is equal to two times of

(55)
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} h(i\lambda) \sum_{\gamma:\ \text{hyperbolic}} l(C_\gamma)j(\gamma)^{-1}D(\gamma)^{-1}\left(\overline{\chi_{\sigma_+}(m_\gamma)} - \overline{\chi_{\sigma_-}(m_\gamma)}\right) e^{-i\lambda l(C_\gamma)}\,d\lambda + \frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty} h(i\lambda)\Phi(i\lambda)\,d\lambda$$
$$= \left(\sum_{\gamma:\ \text{hyperbolic}} l(C_\gamma)j(\gamma)^{-1}D(\gamma)^{-1}\left(\overline{\chi_{\sigma_+}(m_\gamma)} - \overline{\chi_{\sigma_-}(m_\gamma)}\right)\frac{1}{2\pi}\int_{-\infty}^{\infty} h(i\lambda)e^{-i\lambda l(C_\gamma)}d\lambda + \frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty} h(i\lambda)\Phi(i\lambda)d\lambda\right).$$

By (54), (55) and the Selberg trace formula applied to $h(i\lambda)$, we have

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} h(\lambda)R(\lambda)\, d\lambda = 0$$

for any odd holomorphic $h(s)$ satisfying suitable growth conditions. Since $R(s)$ is an entire function, this implies that $R(s) = 0$ over $\mathbb{C}$. Hence we have the equality

$$\frac{d}{ds}\log\left(Z_H^o(n+s)Z_H^o(n-s)\right) \;=\; -d(\sigma_+)\Phi(s)$$

for any $s \in \mathbb{C}$. Finally, recalling the equalities

$$\begin{aligned}\Phi(s) &= \frac{d}{ds}\Big(\log\det C_+(s) - \log\det C_-(s)\Big),\\ \eta(\mathcal{D}) &= \frac{1}{\pi i}\log Z_H^o(n),\end{aligned}$$

we get the following theorem.

THEOREM 7.4. *We have*

$$Z_H^o(n+s)Z_H^o(n-s) = \exp\left(2\pi i\eta(\mathcal{D})\right)\left(\frac{\det C_+(s)C_-(0)}{\det C_-(s)C_+(0)}\right)^{-d(\sigma_+)} \quad \text{for} \quad s \in \mathbb{C}.$$

Remark 7.5. In the above equality, $Z_H^o(n+s)Z_H^o(n-s)$ has zeros at $q_+, -q_+$ of orders $d(\sigma_+)b_+$ for $\mathrm{Re}(q_+) < 0$ and poles at $q_-, -q_-$ of orders $d(\sigma_+)b_-$ for $\mathrm{Re}(q_-) < 0$. These zeros and poles coincide with the zeros and poles of $\left(\frac{\det C_+(s)C_-(0)}{\det C_-(s)C_+(0)}\right)^{-d(\sigma_+)}$.

8. Cusp contributions for regularized determinants and functional equations. In this section, we compute the unipotent factor in the relation between the regularized determinant and the Selberg zeta function of even type. We also derive the functional equation for the regularized determinant and the Selberg zeta function of even type where the unipotent factor plays a nontrivial role. This equation is the even type counterpart of the functional equation for the eta invariant and the Selberg zeta function of odd type proved in Section 7.

We define the zeta function with the factor e^{-ts^2} for a positive real number s by

$$\zeta_{\mathcal{D}^2}(z,s) := \frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}\,\mathrm{Tr}\left(e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa} e^{-t\mathcal{D}_0^2(i)}\right)\, e^{-ts^2}\, dt.$$

Note that $\zeta_{\mathcal{D}^2}(z,0) = \zeta_{\mathcal{D}^2}(z)$ if the kernel of $\mathcal{D}^2$ is trivial. We proved that $\zeta_{\mathcal{D}^2}(z)$ is regular at $z = 0$ in Section 6. Due to the factor s^2, the continuous spectrum of $\mathcal{D}^2 + s^2$ has minimum s^2 and the large time part of the zeta function $\zeta_{\mathcal{D}^2}(z,s)$ does not produce any poles. Therefore $\zeta_{\mathcal{D}^2}(z,s)$ is regular at $z = 0$. We define the regularized determinant of $\mathcal{D}^2 + s^2$ by

$$\operatorname{Det}(\mathcal{D}^2, s) := \exp\left(- \zeta'_{\mathcal{D}^2}(0,s)\right)$$

for a positive real number s. Now we observe that $\zeta_{\mathcal{D}^2}(0,s) = 0$ by the results of Section 6. Then we have

$$\frac{d}{dz}\bigg|_{z=0} \zeta_{\mathcal{D}^2}(z,s) \;=\; \int_0^\infty t^{z-1} \operatorname{Tr}(e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa} e^{-t\mathcal{D}_0^2(i)})\, e^{-ts^2} dt \bigg|_{z=0}.$$

Recall that

$$\begin{aligned}(56)\quad & \operatorname{Tr}\left(e^{-t\mathcal{D}^2} - \sum_{i=1}^{\kappa} e^{-t\mathcal{D}_0^2(i)} \right) \\ &= \sum_{\lambda_k} e^{-t\lambda_k^2} - \frac{d(\sigma_+)}{4\pi} \int_{-\infty}^{\infty} e^{-t\lambda^2} \operatorname{tr}(C_\Gamma(\sigma_+, -i\lambda) C'_\Gamma(\sigma_+, i\lambda))\, d\lambda \\ &= I_\Gamma(K_t^e) + H_\Gamma(K_t^e) + U_\Gamma(K_t^e),\end{aligned}$$

and put

$$\begin{aligned} I(z,s) &:= \int_0^\infty t^{z-1} I_\Gamma(K_t^e)\, e^{-ts^2} dt, \\ H(z,s) &:= \int_0^\infty t^{z-1} H_\Gamma(K_t^e)\, e^{-ts^2} dt, \\ U(z,s) &:= \int_0^\infty t^{z-1} U_\Gamma(K_t^e)\, e^{-ts^2} dt. \end{aligned}$$

Then we have

$$(57)\qquad \frac{1}{2s}\frac{d}{ds} \log \operatorname{Det}(\mathcal{D}^2, s) = I(1,s) + H(1,s) + U(1,s).$$

Now we want to find $Z_I(s), Z_H^e(s), Z_U(s)$ such that

$$I(1,s) = \frac{1}{2s}\frac{d}{ds} \log Z_I(s), \quad H(1,s) = \frac{1}{2s}\frac{d}{ds} \log Z_H^e(s), \quad U(1,s) = \frac{1}{2s}\frac{d}{ds} \log Z_U(s).$$

First,we recall that the Selberg zeta function of even type,

(58)

$$Z_H^e(s) := \exp\left(- \sum_{\sigma\pm} \sum_{\gamma:\ \text{hyperbolic}} j(\gamma)^{-1} |\det(\operatorname{Ad}(a_\gamma m_\gamma)^{-1} - I|_{\mathfrak{n}})|^{-1} \overline{\chi_\sigma(m_\gamma)} e^{-sl(C_\gamma)} \right)$$

is defined for $\mathrm{Re}(s) \gg 0$ and we will show that this has a meromorphic extension over $\mathbb{C}$. We use (9) and elementary equality $\int_0^\infty \frac{1}{\sqrt{4\pi t}} \exp(-(\frac{r^2}{4t}+ts^2))dt = \frac{1}{2s}e^{-rs}$ to see

$$H(1,s) = \frac{1}{2s}\frac{d}{ds}\log Z_H^e(s+n). \tag{59}$$

It follows from Corollary 4.3 that $U(z,s)$ is the Mellin transform with factor e^{-ts^2} of the following terms

$$\frac{2}{2\pi}\int_{-\infty}^{\infty} e^{-t\lambda^2}\left(P_U(\lambda) - \kappa\frac{d(\sigma_\pm)}{2}\left(\psi\left(i\lambda+\frac{1}{2}\right)+\psi\left(-i\lambda+\frac{1}{2}\right)\right)\right)d\lambda \tag{60}$$

where $P_U(\lambda)$ is an even polynomial of degree $(2n-4)$. We deal with the term $\psi(\pm i\lambda + \frac{1}{2})$ using the Cauchy integral formula

$$\begin{aligned}\frac{1}{2\pi}\int_0^\infty\int_{-\infty}^\infty e^{-t(\lambda^2+s^2)}\psi\left(\pm i\lambda+\frac{1}{2}\right)d\lambda\, dt \\ = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{1}{\lambda^2+s^2}\psi\left(\pm i\lambda+\frac{1}{2}\right)d\lambda = \frac{1}{2s}\psi\left(s+\frac{1}{2}\right).\end{aligned}$$

For the part $P_U(\lambda)$ in (60), we have

$$\frac{1}{2s}\frac{d}{ds}\log\exp\left(\int_0^s P_U(i\lambda)\,d\lambda\right) = \frac{1}{2\pi}\int_0^\infty e^{-ts^2}\int_{-\infty}^\infty e^{-t\lambda^2}P_U(\lambda)\,d\lambda\,dt$$

by Lemma 3 in [7]. We define

$$Z_U(s) := \Gamma\left(s+\frac{1}{2}\right)^{-2\kappa d(\sigma_+)}\exp\left(2\int_0^s P_U(i\lambda)\,d\lambda\right).$$

Then this satisfies

$$U(1,s) = \frac{1}{2s}\frac{d}{ds}\log Z_U(s). \tag{61}$$

For $I(z,s)$, we can treat this as for $P_U(\lambda)$ since $p(\sigma_+,\lambda) = p(\sigma_-,\lambda)$ is an even polynomial of λ. Hence, we can see

$$Z_I(s) := \exp\left(2\int_0^s P_I(i\lambda)\,d\lambda\right) \tag{62}$$

where $P_I(\lambda) = 2\pi\mathrm{Vol}(\Gamma\backslash G)p(\sigma_+,\lambda)$. By (57), (59), (61) and (62), we have the following theorem

THEOREM 8.1. *The following equality holds for any $s \in \mathbb{C}$,*

$$\text{(63) } \operatorname{Det}(\mathcal{D}^2, s) = C\, Z_H^e(s+n)\Gamma\left(s+\frac{1}{2}\right)^{-2\kappa d(\sigma_+)} \exp\left(2\int_0^s P_I(i\lambda) + P_U(i\lambda)\, d\lambda\right)$$

where C is a constant and $P_I(\lambda) = 2\pi \operatorname{Vol}(\Gamma\backslash G) p(\sigma_+, \lambda)$.

Proof. A priori, the equality (63) holds for a real number $s \gg 0$ since $Z_H^e(s+n)$ is defined only for $\operatorname{Re}(s) \gg 0$. But the right side of (63) gives the meromorphic extension over $\mathbb{C}$ by Proposition 8.3, and this also gives the meromorphic extension of the left side of (63) over $\mathbb{C}$. □

Remark 8.2. Let us remark that $\operatorname{Det}(\mathcal{D}^2, s) \neq \operatorname{Det}(\mathcal{D}^2, -s)$ as a meromorphic function over $\mathbb{C}$. This is because the equality (63) holds for a real number $s \gg 0$ *a priori* and the meromorphic extension of $\operatorname{Det}(\mathcal{D}^2, s)$ is given by the right side of (63), which does not satisfy this property.

Now let us show that $Z_H^e(s)$ has the meromorphic extension over $\mathbb{C}$. We consider an even smooth function $g(|u|)$ where $g(u)$ is the given function in Section 7. We set

$$H_s(\sigma_\pm, \lambda) = \int_{-\infty}^{\infty} g(|u|)e^{-s|u|}e^{i\lambda u}du$$

for a complex parameter s. Then integration by parts gives

$$\text{(64)} \quad H_s(\sigma_\pm, \lambda) = \frac{1}{s-i\lambda}\int_0^{\infty} g'(u)e^{-(s-i\lambda)u}du + \frac{1}{s+i\lambda}\int_0^{\infty} g'(u)e^{-(s+i\lambda)u}du.$$

Let us apply the Selberg trace formula to the one parameter family of functions f_s on G with $\operatorname{Re}(s) \gg 0$, such that $\hat{f}_s(\sigma_\pm, i\lambda) = H_s(\sigma_\pm, \lambda)$, $\hat{f}_s(\sigma, i\lambda) = 0$ if $\sigma \neq \sigma_\pm$. Then we get

$$\begin{aligned}
\text{(65)} \quad & \sum_{\lambda_j \in \sigma_p^+} H_s(\sigma_+, \lambda_j) + \sum_{\lambda_j \in \sigma_p^-} H_s(\sigma_-, \lambda_j) \\
& \quad - \frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty} H_s(\sigma_+, \lambda)\operatorname{tr}(C_\Gamma(\sigma_+, -i\lambda)C'_\Gamma(\sigma_+, i\lambda))\, d\lambda \\
& = \sum_{\gamma:\ \text{hyperbolic}} l(C_\gamma)j(\gamma)^{-1}D(\gamma)^{-1}\left(\overline{\chi_{\sigma_+}(m_\gamma)} + \overline{\chi_{\sigma_-}(m_\gamma)}\right)e^{-sl(C_\gamma)} \\
& \quad - \frac{\kappa d(\sigma_+)}{2\pi}\int_{-\infty}^{\infty} H_s(\sigma_+, \lambda)\left(\psi\left(i\lambda + \frac{1}{2}\right) + \psi\left(-i\lambda + \frac{1}{2}\right)\right)d\lambda \\
& \quad + \frac{2}{2\pi}\int_{-\infty}^{\infty} H_s(\sigma_+, \lambda)(P_I(\lambda) + P_U(\lambda))\, d\lambda.
\end{aligned}$$

Let us recall the equality

$$\frac{d}{ds}\log Z_H^e(s+n) = \sum_{\gamma:\ \text{hyperbolic}} l(C_\gamma)j(\gamma)^{-1}D(\gamma)^{-1}\left(\overline{\chi_{\sigma_+}(m_\gamma)} + \overline{\chi_{\sigma_-}(m_\gamma)}\right)e^{-sl(C_\gamma)},$$

and now we investigate the other terms in (65) to get the meromorphic extension of $\frac{d}{ds}\log Z_H^e(s+n)$ over $\mathbb{C}$ and to determine its poles.

Discrete eigenvalue term. Using (64) as before, we can see that $H_s(\sigma_\pm, \lambda_j)$ has a meromorphic extension over $\mathbb{C}$ and has the simple poles at $i\lambda_j$ and $-i\lambda_j$ for $\lambda_j \in \sigma_p^\pm$ with the residue m_j where m_j is the multiplicity of λ_j.

Scattering term. Now we consider the scattering term

$$-\frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty} H_s(\sigma_+, \lambda)\ \mathrm{tr}(C_\Gamma(\sigma_+, -i\lambda)C'_\Gamma(\sigma_+, i\lambda))\ d\lambda.$$

As in the previous case, we can show that the function

$$\Psi(z) := \mathrm{tr}\left(C_\Gamma(\sigma_+, -z)\frac{d}{dz}C_\Gamma(\sigma_+, z)\right)$$

has the following form over $\mathbb{C}$:

$$\Psi(z) = \sum_{\mathrm{Re}(q_k)<0} -\frac{2\,\mathrm{Re}(q_k)}{(z-q_k)(z+\bar{q}_k)} + \log\ p$$

for some constant p. Here the sum is taken over the set of poles of $\det C_\Gamma(\sigma_+, z)$. We repeat the method in Section 7 to prove that the scattering term has a meromorphic extension over $\mathbb{C}$ and has poles at q_k for $\mathrm{Re}(q_k) < 0$ with residues $d(\sigma_+)b_k$. As before b_k denotes the order of the pole of $\det C_\Gamma(\sigma_+, z)$ at q_k.

Identity and Unipotent term. It follows from proposition 3.9 in [8] that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} H_s(\sigma_+, \lambda)(P_I(\lambda) + P_U(\lambda))\ d\lambda = 0.$$

We now turn our attention to the other unipotent terms. As in Proposition 3.7 of [8], we use the Cauchy integral formula to get

$$\frac{\kappa d(\sigma_+)}{2\pi}\int_{-\infty}^{\infty} H_s(\sigma_+, \lambda)\left(\psi\left(i\lambda + \frac{1}{2}\right) + \psi\left(-i\lambda + \frac{1}{2}\right)\right)d\lambda = 2\kappa d(\sigma_+)\psi\left(s + \frac{1}{2}\right)$$

for $\mathrm{Re}(s) \gg 0$ and the right side gives us the meromorphic extension over $\mathbb{C}$ of the left side.

Considering the equality (65) and the above analysis of other terms, we conclude:

PROPOSITION 8.3. *The Selberg zeta function of even type $Z_H^e(s)$ has a meromorphic extension over $\mathbb{C}$.*

Remark 8.4. The zeros of $Z_H^e(s+n)\Gamma(s+\frac{1}{2})^{-2\kappa d(\sigma_+)}$ are located at $i\lambda_j$, $-i\lambda_j$ for $\lambda_j \in \sigma_p^{\pm}$, and at q_k for $\mathrm{Re}(q_k) < 0$ and their orders are m_j, $d(\sigma_+)b_k$.

We now prove the functional equation for $\mathrm{Det}(\mathcal{D}^2, s)$ and $Z_H^e(s)$. First let us define

$$\mathcal{Z}(s) := \frac{d}{ds}\log Z_H^e(s+n) - 2\kappa d(\sigma_+)\psi\left(s+\frac{1}{2}\right)$$

and we can see that

$$R(s) := \mathcal{Z}(s) + \mathcal{Z}(-s) + d(\sigma_+)\Psi(s)$$

is an even entire function of s. Now let $h(s)$ be an even function which decreases sufficiently rapidly as $\mathrm{Im}(s) \to \infty$ in the strip $\{ s \in \mathbb{C} \mid |\mathrm{Re}(s)| < n+\epsilon,\ \epsilon > 0 \}$. We follow Section 7 and consider the contour integral

$$\mathcal{L}_T := \frac{1}{2\pi i}\int_{L_T} h(s)\mathcal{Z}(s)\,ds$$

where L_T is the rectangle with the corners $a+iT, a-iT, -a+iT, -a-iT$ with $n < a < n+\epsilon$. Then we have

$$\lim_{T\to\infty}\mathcal{L}_T = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} h(s)(\mathcal{Z}(s)+\mathcal{Z}(-s))\,ds + \frac{2}{2\pi i}\int_{-a+i\infty}^{-a-i\infty} h(s)\mathcal{Z}(s)\,ds.$$

The Cauchy integral theorem gives

$$\lim_{T\to\infty}\mathcal{L}_T = 2\sum_{\lambda_j} m_j h(i\lambda_j) + \sum_{-a<\mathrm{Re}(q_k)<0} d(\sigma_+)b_k h(q_k). \tag{66}$$

The simple poles at q_k of $\mathcal{Z}(s)$ in the strip between $\mathrm{Re}(s) = -a$ and $\mathrm{Re}(s) = 0$ have residues $d(\sigma_+)b_k$ and give the equality

$$\frac{1}{2\pi i}\int_{-a+i\infty}^{-a-i\infty} h(s)\mathcal{Z}(s)ds = \frac{1}{2\pi i}\int_{i\infty}^{-i\infty} h(\lambda)\mathcal{Z}(\lambda)d\lambda + \sum_{-a<\mathrm{Re}(q_k)<0} d(\sigma_+)b_k h(q_k). \tag{67}$$

Similarly, the simple poles at $-q_k$ of $\mathcal{Z}(-s)$ in a strip between $\mathrm{Re}(s) = a$ and

$\mathrm{Re}(s) = 0$ have residues $-d(\sigma_+)b_k$ and provide us with

$$(68)\quad \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} h(s)(\mathcal{Z}(s)+\mathcal{Z}(-s))\,ds = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} h(\lambda)(\mathcal{Z}(\lambda)+\mathcal{Z}(-\lambda))\,d\lambda + \sum_{0<\mathrm{Re}(-q_k)<a} -d(\sigma_+)b_k h(-q_k).$$

We use (66), (67) and (68) to obtain

$$(69)\quad 2\sum_{\lambda_j} m_j h(i\lambda_j) = 2\frac{1}{2\pi i}\int_{i\infty}^{-i\infty} h(\lambda)(\mathcal{Z}(\lambda) + \frac{d(\sigma_+)}{2}\Psi(\lambda))\,d\lambda + \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} h(\lambda)(\mathcal{Z}(\lambda) + \mathcal{Z}(-\lambda) + d(\sigma_+)\Psi(\lambda))\,d\lambda.$$

Replacing λ by $i\lambda$ in the first term of the right side of the equality (69), we see that this term is equal to

$$(70)\quad 2\sum_{\gamma:\ \text{hyperbolic}} l(C_\gamma)j(\gamma)^{-1}D(\gamma)^{-1}\overline{(\chi_{\sigma_+}(m_\gamma)+\chi_{\sigma_-}(m_\gamma))}\frac{1}{2\pi}\int_{-\infty}^{\infty} h(i\lambda)e^{-i\lambda l(C_\gamma)}d\lambda - 2\frac{1}{2\pi}\int_{-\infty}^{\infty} h(i\lambda)(2\kappa d(\sigma_+)\psi(i\lambda+\frac{1}{2}))\,d\lambda + 2\frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty} h(i\lambda)\Psi(i\lambda)d\lambda.$$

By (69), (70) and the Selberg trace formula applied to $h(i\lambda)$, we have

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} h(\lambda)(R(\lambda) + 4P_I(-i\lambda) + 4P_U(-i\lambda))\,d\lambda = 0$$

for any even holomorphic $h(s)$ satisfying suitable growth conditions. Since

$$R(s) + 4P_I(-is) + 4P_U(-is) = R(s) + 4P_I(is) + 4P_U(is)$$

is an entire function, we have

$$\mathcal{Z}(s) + \mathcal{Z}(-s) + 4P_I(is) + 4P_U(is) = -d(\sigma_+)\Psi(s)$$

over $\mathbb{C}$. Now we get the equality

$$\frac{d}{ds}\log\left(Z_H^e(n+s)\Gamma\left(s+\frac{1}{2}\right)^{-2\kappa d(\sigma_+)}\right) + 4P_I(is) + 4P_U(is) = \frac{d}{ds}\log\left(Z_H^e(n-s)\Gamma\left(-s+\frac{1}{2}\right)^{-2\kappa d(\sigma_+)}\right) - d(\sigma_+)\frac{d}{ds}\log(\det C_\Gamma(\sigma_+, s)),$$

which leads to the formula

$$Z^e_H(n+s)\Gamma\left(s+\frac{1}{2}\right)^{-2\kappa d(\sigma_+)}\exp\left(4\int_0^s P_I(i\lambda)+P_U(i\lambda)\,d\lambda\right) = (\det C_\Gamma(\sigma_+,s))^{-d(\sigma_+)}\ Z^e_H(n-s)\Gamma\left(-s+\frac{1}{2}\right)^{-2\kappa d(\sigma_+)}. \tag{71}$$

This equality holds *a priori* up to constant c. As in the proof of Lemma 4.3 in [8], we multiply s^{-2m_0} to both sides of (71) where m_0 is the multiplicity of the zero eigenvalue of $\mathcal{D}$. Doing this removes the spectral zero of $Z^e_H(n\pm s)$ at $s=0$. If we compare the remaining parts at $s=0$, we can see that the constant c equals 1. From (63), we have

$$\mathrm{Det}(\mathcal{D}^2,s)^2 = C^2\,Z^e_H(s+n)^2\Gamma\left(s+\frac{1}{2}\right)^{-4\kappa d(\sigma_+)}\exp\left(4\int_0^s P_I(i\lambda)+P_U(i\lambda)\,d\lambda\right).$$

Finally, we combine (71) and this equality to get

THEOREM 8.5. *For any $s\in\mathbb{C}$, we have*

$$\mathrm{Det}(\mathcal{D}^2,s)^2 = C^2\,(\det C_\Gamma(\sigma_+,s))^{-d(\sigma_+)}\,Z^e_H(n+s)Z^e_H(n-s) \times\left(\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(-s+\frac{1}{2}\right)\right)^{-2\kappa d(\sigma_+)}. \tag{72}$$

Remark 8.6. The right side of (72) has the zeros at $i\lambda_j$, $-i\lambda_j$ of order $2m_j$ for $\lambda_j\in\sigma_p^{\pm}$, and at q_k of order $2d(\sigma_+)b_k$ for $\mathrm{Re}(q_k)<0$. These are the zeros of $\mathrm{Det}(\mathcal{D}^2,s)^2$ as we expected.

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