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On the gluing problem for the spectral invariants of Dirac operators

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Abstract

In this paper, we solve the gluing problem for the ζ-determinant of a Dirac Laplacian. To do so, we develop a new approach to solve such problems which relies heavily on the theory of elliptic boundary problems, the analysis of the resolvent of the Dirac operator, and the introduction of an auxiliary model problem. Moreover, as a byproduct of our approach we obtain a new gluing formula for the eta invariant au gratis.

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1. Introduction

Over the past several years there has been intense research activity in understanding the behavior of geometric and spectral invariants of Dirac-type operators under gluing, or surgery, of the underlying Riemannian manifold. This explosion has resulted in the search for gluing or pasting formulas for these invariants and has been motivated both

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by geometers in regards to, for instance, applications to the Ray–Singer conjecture concerning the equality of torsion invariants, index theory on manifolds with corners and gluing formulas for Dirac determinant line bundles, and by mathematical physicists because of the rôle of these invariants in Donaldson, Floer, and Seiberg–Witten theory and especially in the development of topological quantum field theory where pasting laws for ζ-determinants are required. For recent reviews, see Mazzeo and Piazza [33] and Scott and Wojciechowski [41].

However, the gluing formula for the ζ-determinant of a Dirac Laplacian has remained an open question due to the nonlocal nature of this invariant. The purpose of this paper is to solve this gluing problem. To do so, we develop a new technique to attack such problems by using the theory of elliptic boundary problems, the analysis of the resolvent of the Dirac operator, and the introduction of an auxiliary model problem where the gluing problem can be solved explicitly [28]. This technique can be adapted to more general cases like noncompact manifolds or the nonproduct situation near the cutting hypersurface where, in the forthcoming papers [27,29,30], we shall investigate similar gluing problems.

We begin with a brief history and rough description of gluing problems; in a moment we shall make these notions precise. The basic statement is as follows: given a partitioned compact manifold $M = M_- \cup M_+$ into manifolds with boundaries, describe the geometric and spectral invariants of Dirac operators on M in terms of the invariants on $M_\pm$ with suitable boundary conditions. Here, we consider the index as a geometric invariant and the eta invariant and ζ-determinant as spectral invariants. The gluing problem for the index was the first to be solved. This was settled by Atiyah et al. [1], and the solution used the local nature of the index. The *Bojarski Conjecture*, later proved by Booß-Bavnbek and Wojciechowski [4], gives a general gluing formula for the index in terms of the Fredholm index of the pair of Cauchy data spaces from $M_\pm$. Later, we will see that Cauchy data spaces play significant rôles in the solutions of each of the gluing problems. Next, the gluing problem for the eta invariant was solved. The main difficultly in this case has to do with the nonlocal nature of the eta invariant, in contrast to the local nature of the index. However, the variation of the eta invariant is local, and because of this locality a variety of formulas and proofs for the gluing problem have been found (many modulo $\mathbb{Z}$), see for instance, Brüning and Lesch [5], Bunke [6], Dai and Freed [10], Douglas and Wojciechowski [11], Hassell et al. [18], Kirk and Lesch [21], Mazzeo and Melrose [32], Müller [34], Park and Wojciechowski [37], Wojciechowski [47,48]; see the survey articles by Bleecker and Booß-Bavnbek [3] and Mazzeo and Piazza [33] for more on this topic. Of these solutions, the one by Kirk and Lesch [21] is the most complete and general and, as with Booß-Bavnbek and Wojciechowski's [4] solution to the index problem, involves the two Cauchy data spaces from $M_\pm$.

Last in the chain of invariants is the ζ-determinant. Because of the highly nonlocal nature of the ζ-determinant and its variation, the gluing problem for the ζ-determinant has been the most difficult part of the gluing problems to solve. Nonetheless, the gluing problem for the ζ-determinant of Laplace-type operators with *local* boundary conditions was solved by Burghelea et al. [7] and has been further extended by Carron [9], Hassell [17], Hassell and Zelditch [19], Lee [22], Loya and Park [25], Vishik [46], and many others. Compounding the nonlocal nature of the ζ-determinant and its variation with the technical aspects inherent with the *global* pseudodifferential boundary problems required for Dirac-type operators, the gluing problem for the ζ-determinant of Dirac-type operators has remained an open problem. The purpose of this paper is to solve this gluing problem under general pseudodifferential boundary conditions and to develop a new method to attack such problems. As with Booß-Bavnbek and Wojciechowski's [4] solution to the index problem and Kirk and Lesch's [21] solution to the eta problem, our solution involves the two Cauchy data spaces from $M_{\pm}$. As we will see later, this is because the gluing problems for the eta invariant and the ζ-determinant are not entirely separate problems, but are really just two aspects of one problem—*the phase and modulus* of the same global data defined by the two Cauchy data spaces from $M_{\pm}$.

We now describe our situation more precisely. Let $\mathcal{D}$ be a self-adjoint Dirac-type operator acting on $C^{\infty}(M, S)$ where M is a closed compact Riemannian manifold of arbitrary dimension and S is a Clifford bundle over M. We decompose the closed manifold M into two submanifolds M_-, M_+ with a common boundary Y such that

$$M = M_- \cup M_+, \quad \partial M_- = \partial M_+ = Y.$$

We also assume throughout this paper that the Riemannian metric of M and the Hermitian metric of S are of product type over a tubular neighborhood $N = [-1, 1] \times Y$ of Y where the Dirac operator takes the product form

$$\mathcal{D} = G(\partial_u + D_Y).$$

Here, G is an endomorphism of $S_0 := S|_Y$ and D_Y is a Dirac-type operator over Y satisfying $G^2 = -\mathrm{Id}$ and $D_Y G = -G D_Y$. Recall that if $\{\lambda_k\}$ are the eigenvalues of $\mathcal{D}$, then the eta function of $\mathcal{D}$ is defined by

$$\eta_{\mathcal{D}}(s) = \sum_{\lambda_k \neq 0} \frac{\mathrm{sign}(\lambda_k)}{|\lambda_k|^s}$$

and the zeta function of $\mathcal{D}^2$ is defined by

$$\zeta_{\mathcal{D}^2}(s) = \sum_{\lambda_k \neq 0} \lambda_k^{-2s},$$

both of which are defined a priori for $\mathrm{Re}\, s \gg 0$ and extend to be meromorphic functions on $\mathbb{C}$ that are regular at $s = 0$. The eta invariant $\eta(\mathcal{D})$ is by definition the value $\eta_{\mathcal{D}}(0)$ and the ζ-determinant is by definition

$$\det{}_{\zeta} \mathcal{D}^2 := \exp\Big(- \frac{d}{ds}\Big|_{s=0} \zeta_{\mathcal{D}^2}(s) \Big). \tag{1.1}$$

The eta invariant was introduced in the paper [1] by Atiyah et al. as the boundary correction term in their index formula for manifolds with boundary. The ζ-determinant was introduced by Ray and Singer in the paper [38] on the analytic torsion. Since these papers, eta invariants and ζ-determinants have impacted geometry, topology, and physics in several ways, cf. Singer [45,44] and Hawking [20].

To formulate the gluing problem, we need to introduce boundary conditions. By restriction, $\mathcal{D}$ induces Dirac-type operators $\mathcal{D}_+$ over M_+ and $\mathcal{D}_-$ over M_-. For these operators, we choose orthogonal projections $\mathcal{P}_+$, $\mathcal{P}_-$ over $L^2(Y, S_0)$, that provide us with *well-posed boundary conditions* for $\mathcal{D}_+$, $\mathcal{D}_-$ in the sense of Seeley [43]. Then we obtain Fredholm operators

$$\mathcal{D}_{\mathcal{P}_\pm} : \mathrm{dom}(\mathcal{D}_{\mathcal{P}_\pm}) \to L^2(M_\pm, S),$$

where

$$\mathrm{dom}(\mathcal{D}_{\mathcal{P}_\pm}) := \{\phi \in H^1(M_\pm, S) \mid \mathcal{P}_\pm(\phi|_Y) = 0\}.$$

Amongst these projectors are the *orthogonalized* Calderón projectors $\mathcal{C}_\pm$ [8], which are projectors defined intrinsically as the unique orthogonal projectors onto the closures in $L^2(Y, S_0)$ of the infinite-dimensional *Cauchy data spaces* of $\mathcal{D}_\pm$:

$$\{\phi|_Y \mid \phi \in C^\infty(M_\pm, S),\ \mathcal{D}_\pm\phi = 0\} \subset C^\infty(Y, S_0).$$

In order to define the eta invariant of $\mathcal{D}_{\mathcal{P}_\pm}$ and the ζ-determinant of $\mathcal{D}^2_{\mathcal{P}_\pm}$, we need to restrict to a subclass of projectors; a natural class of such projectors are those in the *smooth*, *self-adjoint Grassmannian* $\mathrm{Gr}^*_\infty(\mathcal{D}_\pm)$, which consists of orthogonal projections $\mathcal{P}_\pm$ with two properties; (1) $\mathcal{P}_\pm - \mathcal{C}_\pm$ are smoothing operators; (2) $G\mathcal{P}_\pm = (\mathrm{Id} - \mathcal{P}_\pm)G$. Examples of such projections are the generalized APS spectral projections $\mathcal{P}_- = \Pi_< + \frac{1+\sigma_1}{2}\Pi_0$, $\mathcal{P}_+ = \Pi_> + \frac{1+\sigma_2}{2}\Pi_0$ where $\Pi_<$, $\Pi_>$, Π_0 are the orthogonal projections onto the eigenspaces of the negative, positive, and zero eigenvalues of D_Y, respectively, and the σ_i's are involutions on $\ker(D_Y)$ anticommuting with G. For $\mathcal{P}_\pm \in \mathrm{Gr}^*_\infty(\mathcal{D}_\pm)$, the eta invariant $\eta(\mathcal{D}_{\mathcal{P}_\pm})$ and ζ-determinant $\det_\zeta \mathcal{D}^2_{\mathcal{P}_\pm}$ can be defined in the same way as in the closed case, see Grubb [14,15], Loya and Park [26] and Wojciechowski [48].

The *gluing problem* for the spectral invariants is to describe the "defects"

$$\eta(\mathcal{D}) - \eta(\mathcal{D}_{\mathcal{P}_+}) - \eta(\mathcal{D}_{\mathcal{P}_-}), \tag{1.2}$$

$$\log \det_\zeta \mathcal{D}^2 - \log \det_\zeta \mathcal{D}^2_{\mathcal{P}_+} - \log \det_\zeta \mathcal{D}^2_{\mathcal{P}_-} \tag{1.3}$$

in terms of recognizable data. These types of problems are also called *surgery*, *pasting*, *or splitting*, *problems of the spectral invariants*. The aim of this paper is to provide complete simultaneous solutions to these gluing problems and to develop a new method to attack such problems.

Before explaining our approach to the gluing problem, we first discuss the choice of the projections $\mathcal{P}_\pm$ imposing the boundary conditions. After Atiyah, Patodi, and Singer introduced the APS spectral projector in their influential paper [1], the APS spectral projector has been used, to some extent, in the gluing formulas for the eta invariant; moreover, most of the formulas hold modulo an integer ambiguity, cf. [3]. This ambiguity was removed by Kirk and Lesch [21] where they formulated their result in terms of the boundary conditions given by the Calderón projectors rather than the APS spectral projectors. This suggests that it is more appropriate to use the Calderón projector instead of the APS spectral projector in regards to the gluing formula for the eta invariant. For the ζ-determinant gluing formula of the Dirac Laplacian, one can also see the need to use Calderón projectors by the work of Park and Wojciechowski [36,37] on the adiabatic decomposition of the ζ-determinant. Here, 'adiabatic decomposition' means to investigate the limit of the ratio of the ζ-determinants of the whole manifold and the decomposed manifolds as the length of the collar $N = [-1, 1] \times Y$ is stretched to infinity. This limiting value is described by a ratio of determinants involving the scattering matrices defined from the manifolds obtained by attaching half infinite cylinders to the decomposed manifolds with boundary. Here, the scattering matrix is the analog of the Calderón projector for manifolds with cylindrical end.

These phenomena lead us to first solve the gluing problem for the ζ-determinant (1.3) with respect to the Calderón projectors $\mathcal{P}_\pm = \mathcal{C}_\pm$ rather than the APS spectral projectors (see Theorem 1.1), and then only after understanding this case, proceed to general $\mathcal{P}_\pm$ (see Theorem 1.2).

Because we choose to work with Calderón projectors and therefore have no explicit form for the heat kernels of $\mathcal{D}^2_{\mathcal{C}_\pm}$, we have to proceed in a different way from the established methods used to derive gluing formula of the eta invariant. As a consequence, we develop a new method which relies on the theory of elliptic boundary problems, the analysis of the resolvents of $\mathcal{D}$ and $\mathcal{D}_{\mathcal{C}_\pm}$, and the introduction of an auxiliary model problem. The basic idea is to introduce a family of operators $K(\lambda)$ over Y defined by the Calderón projectors of $(\mathcal{D}_\pm - \lambda)$ on $M_\pm$. We then connect the operator $K(\lambda)$ to the resolvents $(\mathcal{D} - \lambda)^{-1}$ and $(\mathcal{D}_{\mathcal{C}_\pm} - \lambda)^{-1}$, and then connect the resolvents to the spectral invariants. The operator $K(\lambda)$ describes how the two Cauchy data spaces of $(\mathcal{D}_\pm - \lambda)$ over $M_\pm$ match into the global data of $(\mathcal{D} - \lambda)$ over M, which simultaneously contains the *phase* data describing the eta invariants (1.2) and the *modulus* data describing the ζ-determinants (1.3). From this view point, the gluing problems of the eta invariant and the ζ-determinant simply represent two aspects of one problem. Another new feature of our method is the introduction of an *auxiliary model problem* over the finite cylinder $N = [-1, 1] \times Y$. We consider the corresponding gluing problem and family of operators $K^c(\lambda)$ for the decomposition of N into its two halves $N_- = [-1, 0] \times Y$ and $N_+ = [0, 1] \times Y$. The essence of our approach is to compare the original problem over M with this model problem over N, where the gluing problem can be solved exactly [28]. This enables us to avoid certain *trace class issues* involving the resolvents $(\mathcal{D} - \lambda)^{-1}$ and $(\mathcal{D}_{\mathcal{C}_\pm} - \lambda)^{-1}$ and allows us to derive the gluing formulas with no undetermined constants.

The use of resolvents in the study of spectral invariants can also be found in, for instance, Forman [12] (cf. [23]), Burghelea et al. [7], Scott and Wojciechowski [40], Scott [39], and Loya and Park [25,26]. However, the papers [12,40,39,26] deal with the relative invariant problem, and not the gluing problem, of the spectral invariants with two different boundary conditions over a single manifold with boundary. Hence, there is no gluing feature concerning the mechanism to which the two Cauchy data spaces from $M_\pm$ connect with the global data of the closed manifold M; in contrast, this is the crucial point for the gluing problem. This distinction makes our analysis substantially different from the analysis used for the relative invariant problem. The gluing formula for the ζ-determinant of a Laplace-type operator over M with Dirichlet boundary conditions was considered in [7,25]. In this case, the resolvents of the Laplace-type operators (with Dirichlet boundary condition) over M (and $M_\pm$) are linked through the sum of the Dirichlet to Neumann operators defined over $M_\pm$. However, this operator cannot be applied to the Dirac operator situation where one deals with global pseudodifferential projections and not the local Dirichlet condition.

We now state our main theorem. The Calderón projectors $\mathcal{C}_\pm$ have the matrix forms

$$\mathcal{C}_\pm = \frac{1}{2}\begin{pmatrix} \mathrm{Id} & \kappa_\pm^{-1} \\ \kappa_\pm & \mathrm{Id} \end{pmatrix} \tag{1.4}$$

with respect to $L^2(Y, S_0) = L^2(Y, S^+) \oplus L^2(Y, S^-)$ where $S^\pm$ are the subbundles of S_0 consisting of the $(\pm i)$-eigensections of G. Here, the maps $\kappa_\pm : L^2(Y, S^+) \to L^2(Y, S^-)$ are isometries, so that $U := -\kappa_- \kappa_+^{-1}$ is a unitary operator over $L^2(Y, S^-)$. Moreover, this operator is of Fredholm determinant class. We denote by $\widehat{U}$ the restriction of U to the orthogonal complement of its (-1)-eigenspace. We also put

$$\mathcal{L} := \sum_{k=1}^{h_M} \gamma_0 U_k \otimes \gamma_0 U_k = \sum_{k=1}^{h_M} \langle \cdot, \gamma_0 U_k \rangle_{L^2(Y,s_0)} \gamma_0 U_k,$$

where $h_M = \dim \ker(\mathcal{D})$, γ_0 is the restriction map from M to Y, and $\{U_k\}$ is an orthonormal basis of the kernel of $\mathcal{D}$. Then $\mathcal{L}$ is a positive operator on the finite-dimensional vector space $\gamma_0(\ker(\mathcal{D}))$. The following theorem is the main result of our paper.

Theorem 1.1. *The following ζ-determinant gluing formula holds*:

$$\frac{\det_\zeta \mathcal{D}^2}{\det_\zeta \mathcal{D}^2_{\mathcal{C}_+} \cdot \det_\zeta \mathcal{D}^2_{\mathcal{C}_-}} = 2^{-\zeta_{D_Y^2}(0) - h_Y} (\det \mathcal{L})^{-2} \det_F \left(\frac{2\mathrm{Id} + \widehat{U} + \widehat{U}^{-1}}{4} \right),$$

where $\zeta_{D_Y^2}(s)$ is the ζ-function of D_Y^2, $h_Y = \dim \ker(D_Y)$, and $\det_F$ denotes the Fredholm determinant.

A brief sketch of the proof of Theorem 1.1 is as follows. We adopt the strategy of Forman [12] and Burghelea et al. [7], by introducing a spectral parameter λ and relying heavily on resolvents. For our problem, we consider the aforementioned operator $K(\lambda)$ which depends on the Cauchy data spaces of $(\mathcal{D}_\pm - \lambda)$ on $M_\pm$ and the corresponding operator $K^c(\lambda)$ over the model cylinder. We prove that $K(\lambda)K^c(\lambda)^{-1}$ is of Fredholm determinant class and show that the λ-derivative of $\log \det_F(K(\lambda)K^c(\lambda)^{-1})$ equals a relative trace of resolvents $(\mathcal{D} - \lambda)^{-1}$, $(\mathcal{D}_{\mathcal{C}_\pm} - \lambda)^{-1}$ and the corresponding resolvents over the model cylinder. On the other hand, for purely imaginary λ, the λ-derivative of the log of a ratio of the ζ-determinants of $\mathcal{D}^2 - \lambda^2$, $\mathcal{D}^2_{\mathcal{C}_\pm} - \lambda^2$ and the corresponding operators over the model cylinder can be expressed in terms of this relative trace. Hence, we obtain a relation between $\det_F(K(\lambda)K^c(\lambda)^{-1})$ and a ratio of the ζ-determinants of $\mathcal{D}^2 - \lambda^2$, $\mathcal{D}^2_{\mathcal{C}_\pm} - \lambda^2$ and the corresponding operators over the model cylinder up to an integration constant. We then study the asymptotics of $\det_F(K(\lambda)K^c(\lambda)^{-1})$ as $\Im\lambda \to \pm\infty$ to determine this constant. Finally, we study the asymptotics of $\det_F(K(\lambda)K^c(\lambda)^{-1})$ as $\lambda \to 0$ and combine this with the exact gluing formula over the model cylinder to get the desired formula in Theorem 1.1.

We can generalize the gluing formula stated in Theorem 1.1 in terms of other boundary conditions if we combine Theorem 1.1 with the result in [26]. To explain this generalization, let $\mathcal{P}_1 \in \mathrm{Gr}^*_\infty(\mathcal{D}_-)$ and $\mathcal{P}_2 \in \mathrm{Gr}^*_\infty(\mathcal{D}_+)$. The projections $\mathcal{P}_1$ and $\mathcal{P}_2$ determine maps κ_1 and κ_2 as in (1.4), so we can define unitary operators $U_1 := \kappa_- \kappa_1^{-1}$ and $U_2 := \kappa_2 \kappa_+^{-1}$ over $L^2(Y, S^-)$. As before, we let $\widehat{U_i}$ denote the restriction of U_i to the orthogonal complement of its (-1)-eigenspace. We define the operator $\mathcal{L}_1$ over the finite-dimensional vector space $\mathrm{ran}(\mathcal{C}_-) \cap \mathrm{ran}(\mathrm{Id} - \mathcal{P}_1)$ by

$$\mathcal{L}_1 := -P_1\, G\, \mathcal{R}_-^{-1}\, G\, P_1,$$

where $\mathcal{R}_-$ is the sum of the Dirichlet to Neumann maps on the double of M_-, that was introduced in [7], and P_1 is the orthogonal projection onto $\mathrm{ran}(\mathcal{C}_-) \cap \mathrm{ran}(\mathrm{Id} - \mathcal{P}_1)$. In [26], we prove that $\mathcal{L}_1$ is a positive operator so that $\det \mathcal{L}_1$ is a positive real number. We define $\mathcal{L}_2$ in a similar way. We can now state the general gluing formula for the ζ-determinant.

Theorem 1.2. *The following general gluing formula holds*:

$$\frac{\det_\zeta \mathcal{D}^2}{\det_\zeta \mathcal{D}^2_{\mathcal{P}_1} \cdot \det_\zeta \mathcal{D}^2_{\mathcal{P}_2}} = 2^{-\zeta_{D_Y^2}(0) - h_Y} (\det \mathcal{L})^{-2} \det_F \left(\frac{2\mathrm{Id} + \widehat{U} + \widehat{U}^{-1}}{4} \right)$$
$$\times \prod_{i=1}^{2} (\det \mathcal{L}_i)^{-2} \cdot \det_F \left(\frac{2\mathrm{Id} + \widehat{U_i} + \widehat{U_i}^{-1}}{4} \right)^{-1}.$$

The gluing formula of the ζ-determinant in Theorem 1.2 holds in particular for the generalized APS spectral projectors $\mathcal{P}_1 = \Pi_< + \frac{1+\sigma_1}{2}\Pi_0$, $\mathcal{P}_2 = \Pi_> + \frac{1+\sigma_2}{2}\Pi_0$. The resulting formula is not so simple in comparison to the formula in Theorem 1.1, which reinforces the prominent rôle of the Calderón projector in gluing problems over the APS projector. However, taking generalized APS projectors in Theorem 1.2 and elongating the collar, and using the description of the adiabatic limit of the Cauchy data spaces [35], we get a simple derivation of the aforementioned adiabatic decomposition formula of the ζ-determinant of a Dirac Laplacian presented in [36], [37], which was proved mainly using the Duhamel principle and small eigenvalue analysis through scattering theory (cf. [31]). As we already mentioned, the method we use to attack the ζ-determinant actually solves the gluing problem for the eta invariant au gratis:

Corollary 1.3 (*of proof*). *The following gluing formula holds*:

$$\tilde{\eta}(\mathcal{D}) - \tilde{\eta}(\mathcal{D}_{\mathcal{P}_1}) - \tilde{\eta}(\mathcal{D}_{\mathcal{P}_2}) = \frac{1}{2\pi i}\,\mathrm{Log}\,\mathrm{det}_F\, U_{12} \mod \mathbb{Z},$$

where the tildes denote reduced eta invariants, e.g. $\tilde{\eta}(\mathcal{D}) = (\eta(\mathcal{D}) + \dim\ker(\mathcal{D}))/2$, Log *the principal branch of the logarithm, and* $U_{12} := -\kappa_1\kappa_2^{-1}$.

Remark 1.4. As shown in Section 9, the integer defect can be *identified exactly* in terms of winding numbers involving the fundamental operator $K(\lambda)$, the corresponding operator $K^c(\lambda)$ for the auxiliary model problem, and related operators from the relative invariant problem in [26]; this is a new formulation of the integer defect. The eta formula (with a different right-hand side) was first proved by Kirk and Lesch [21, Theorem 5.10] using techniques from [5,40]. However, the proof of the eta gluing formula we present is distinct from theirs and in our case it is proved 'simultaneously' with the ζ-determinant gluing formula.

The structure of this paper is as follows. In Section 2, we review some material on elliptic boundary problems for Dirac-type operators. In Section 3, we introduce the model problem over the finite cylinder N and compare this with our original problem. In Section 4, we introduce a family of operators $K(\lambda)$ that links the Cauchy data spaces of $(\mathcal{D}_\pm - \lambda)$ with the resolvents $(\mathcal{D} - \lambda)^{-1}$ and $(\mathcal{D}_{\mathcal{C}_\pm} - \lambda)^{-1}$ and we define the corresponding operator $K^c(\lambda)$ for the auxiliary model problem. We also prove an equality between the variation of $\log \mathrm{det}_F\,(K(\lambda)K^c(\lambda)^{-1})$ and the relative traces of the resolvents. In Section 5, we discuss the asymptotic behavior of $\mathrm{det}_F\,(K(\lambda)K^c(\lambda)^{-1})$ for λ near 0. This analysis enables us to determine the contribution $\det \mathcal{L}$ in the formula of the ζ-determinant. In Section 6, we study the limits of the Calderón projectors for $(\mathcal{D}_\pm - \lambda)$ as $\Im\lambda \to \pm\infty$ and we use this in Section 7 to derive asymptotic expansions for $\mathrm{det}_F\,(K(\lambda)K^c(\lambda)^{-1})$ as $\Im\lambda \to \pm\infty$. In Section 8, we express the spectral invariants of $\mathcal{D}$, $\mathcal{D}_{\mathcal{C}_\pm}$, and those on the finite cylinder in terms of $\log \mathrm{det}_F\,(K(\lambda)K^c(\lambda)^{-1})$, so that we can apply the results in the previous sections to prove the gluing formulas of the spectral invariants. Finally, combining all the results established in the previous sections, in Section 9 we prove Theorems 1.1, 1.2, and Corollary 1.3.

2. Elliptic boundary problems for Dirac-type operators

In this section, we review basic material on elliptic boundary problems for Dirac-type operators. We follow the notation in the introduction.

Throughout this paper, we shall fix a union of sectors $\Lambda \subset \mathbb{C}$ of the form

$$\Lambda = \{\lambda \in \mathbb{C} \setminus \{0\} \mid \varepsilon_0 \leqslant \arg \lambda \leqslant \pi - \varepsilon_0 \text{ or } \pi + \varepsilon_0 \leqslant \arg \lambda \leqslant 2\pi - \varepsilon_0\}, \tag{2.1}$$

where $0 < \varepsilon_0 < \pi/2$. For $\lambda \in \Lambda$, we define

$$\mathcal{D}(\lambda) = \mathcal{D} - \lambda : H^1(M, S) \to L^2(M, S),$$

then its inverse operator $\mathcal{D}(\lambda)^{-1}$ is well-defined. We denote the restriction of $\mathcal{D}(\lambda)$ over $M_\pm$ by $\mathcal{D}_\pm(\lambda)$. We denote the trace, or restriction map to $Y_\varepsilon = \{\varepsilon\} \times Y \subset M$ by γ_ε. Then γ_ε is a well-defined map

$$\gamma_\varepsilon : H^k(M, S) \to H^{k-\frac{1}{2}}(Y_\varepsilon, S_0)$$

for $k > \frac{1}{2}$. The *Calderón projectors* of $\mathcal{D}_\pm(\lambda)$ are defined by

$$P_\pm(\lambda) = \pm\gamma_{0^\pm}\mathcal{D}(\lambda)^{-1}\gamma_0^* G : L^2(Y, S_0) \to L^2(Y, S_0), \tag{2.2}$$

where $\gamma_{0^\pm} = \lim_{\varepsilon\to 0^\pm} \gamma_\varepsilon$ and γ_0^* is the adjoint map of γ_0 at $\{0\} \times Y$. In [42,13], it is proved that $P_\pm(\lambda)$ are pseudodifferential projections, their images coincide with the closures in $L^2(Y, S_0)$ of *Cauchy data spaces* of $\mathcal{D}_\pm(\lambda)$:

$$\mathcal{H}_\pm(\lambda) = \left\{\gamma_{0^\pm}\phi_\pm \mid \phi_\pm \in C^\infty(M_\pm, S),\ \mathcal{D}_\pm(\lambda)\phi_\pm = 0 \text{ over } M_\pm \setminus Y\right\}$$

and the following equality holds:

$$P_+(\lambda) + P_-(\lambda) = \mathrm{Id}. \tag{2.3}$$

With $\mathcal{C}_\pm$ denoting the unique *orthogonalized* Calderón projectors, which are by definition the (unique) orthogonal projectors onto the closures of the Cauchy data spaces $\mathcal{H}_\pm(0)$, we define

$$\begin{aligned} S_\pm(\lambda) &= \mathcal{C}_\pm P_\pm(\lambda), \\ S_\pm(\lambda)^{-1} &= P_\pm^{\mathrm{o}}(\lambda)[\mathcal{C}_\pm P_\pm^{\mathrm{o}}(\lambda) + (\mathrm{Id} - \mathcal{C}_\pm)(\mathrm{Id} - P_\pm^{\mathrm{o}}(\lambda))]^{-1}\mathcal{C}_\pm, \end{aligned}$$

where $P^{\text{o}}_{\pm}(\lambda)$ are the unique orthogonal projections onto the closures of the Cauchy data spaces $\mathcal{H}_{\pm}(\lambda)$ given by (see [2,4])

$$P^{\text{o}}_{\pm}(\lambda) := P_{\pm}(\lambda)P^*_{\pm}(\lambda)\big(P_{\pm}(\lambda)P^*_{\pm}(\lambda) + (\mathrm{Id} - P^*_{\pm}(\lambda))(\mathrm{Id} - P_{\pm}(\lambda))\big)^{-1}.$$

Then $S_{\pm}(\lambda)$ and $S_{\pm}(\lambda)^{-1}$ satisfy the following equations [13]:

$$S_{\pm}(\lambda)S_{\pm}(\lambda)^{-1} = \mathcal{C}_{\pm}, \quad S_{\pm}(\lambda)^{-1}S_{\pm}(\lambda) = P_{\pm}(\lambda). \tag{2.4}$$

Using these formulas and that $S_{\pm}(\lambda)$ are holomorphic in $\lambda \in \Lambda$, it is straightforward to show that $S_{\pm}(\lambda)^{-1}$ are holomorphic functions of $\lambda \in \Lambda$.

For the Dirac-type operators $\mathcal{D}_{\pm}(\lambda)$, we impose the boundary conditions given by $\mathcal{C}_{\pm}$ and denote the resulting operators by

$$\mathcal{D}_{\mathcal{C}_{\pm}}(\lambda) : \mathrm{dom}(\mathcal{D}_{\mathcal{C}_{\pm}}(\lambda)) \to L^2(M, S),$$

where

$$\mathrm{dom}(\mathcal{D}_{\mathcal{C}_{\pm}}(\lambda)) := \{\phi \in H^1(M_{\pm}, S) \,|\, \mathcal{C}_{\pm}\gamma_0\phi = 0\}.$$

We define

$$\mathcal{D}_{\pm}(\lambda)^{-1} = r_{\pm}\mathcal{D}(\lambda)^{-1}e_{\pm},$$

where $r_{\pm} : H^1(M, S) \to H^1(M_{\pm}, S)$ are the restriction maps from M to $M_{\pm}$ and $e_{\pm} : L^2(M_{\pm}, S) \to L^2(M, S)$ are the extension maps by zero out of $M_{\pm}$. The *Poisson operators* of $\mathcal{D}_{\mathcal{C}_{\pm}}(\lambda)$ are defined by

$$\mathcal{K}_{\mathcal{C}_{\pm}}(\lambda) = \mathcal{K}_{\pm}(\lambda)S_{\pm}(\lambda)^{-1}, \tag{2.5}$$

where $\mathcal{K}_{\pm}(\lambda) = \pm\mathcal{D}_{\pm}(\lambda)^{-1}\gamma_0^*G$. These operators $\mathcal{K}_{\mathcal{C}_{\pm}}(\lambda)$ satisfy

$$(\mathcal{D} - \lambda)\mathcal{K}_{\mathcal{C}_{\pm}}(\lambda) = 0, \quad \mathcal{C}_{\pm}\gamma_0\mathcal{K}_{\mathcal{C}_{\pm}}(\lambda) = \mathcal{C}_{\pm}. \tag{2.6}$$

Then the following equality holds [13]:

$$\mathcal{D}_{\mathcal{C}_{\pm}}(\lambda)^{-1} = \mathcal{D}_{\pm}(\lambda)^{-1} - \mathcal{K}_{\mathcal{C}_{\pm}}(\lambda)\mathcal{C}_{\pm}\gamma_0\mathcal{D}_{\pm}(\lambda)^{-1}. \tag{2.7}$$

3. Comparison with the auxiliary model problem

In this section, we compare our original problem with a corresponding *auxiliary model problem* on the finite cylinder N, which is defined as follows.

We consider the restriction of the Dirac-type operator $\mathcal{D}(\lambda)$ to N, whose boundary consists of two components $\{\pm 1\} \times Y$. Recall that $\Pi_>$, $\Pi_<$, and Π_0 denote the orthogonal projections onto the positive, negative, and zero eigenspaces, respectively, of D_Y. By the cobordism invariance of the index [4], we have $\dim\big(\ker(D_Y) \cap C^\infty(Y, S^+)\big) = \dim\big(\ker(D_Y) \cap C^\infty(Y, S^-)\big)$, so we can henceforth fix an involution σ over $\ker(D_Y)$ that anticommutes with G. We then impose boundary conditions $\mathcal{C}^c_- = \Pi_< + \frac{1-\sigma}{2}\Pi_0$ at $\{1\} \times Y$, $\mathcal{C}^c_+ = \Pi_> + \frac{1+\sigma}{2}\Pi_0$ at $\{-1\} \times Y$ for the restriction of $\mathcal{D}(\lambda)$ to N. Let us denote the resulting operator with these boundary conditions by $\mathcal{D}^c(\lambda)$. Then

$$\mathcal{D}^c(\lambda) : \mathrm{dom}(\mathcal{D}^c(\lambda)) \to L^2(N, S),$$

where

$$\mathrm{dom}(\mathcal{D}^c(\lambda)) := \{\phi \in H^1(N, S) \mid \mathcal{C}^c_\pm\big(\gamma_{\mp 1}\phi\big) = 0\}.$$

We denote the restriction of $\mathcal{D}^c(\lambda)$ to $N_- = [-1, 0] \times Y$ by $\mathcal{D}^c_-(\lambda)$ and to $N_+ = [0, 1] \times Y$ by $\mathcal{D}^c_+(\lambda)$. If $\mathcal{K}^c(\lambda) = \mathcal{D}^c(\lambda)^{-1}\gamma_0^* G$, then the usual arguments using the rational symbolic structure of $\mathcal{D}^c(\lambda)^{-1}$ (cf. the proof of Theorem 6.1) show that if $\varphi \in C^\infty(Y, S_0)$, then $\big(\mathcal{K}^c(\lambda)\varphi\big)\big|_{N_\pm} \in \mathrm{dom}(\mathcal{D}^c_\pm(\lambda))$ and is smooth up to each side of Y with at most a jump discontinuity at Y. In particular, we can define the Calderón projectors $P^c_\pm(\lambda)$ of $\mathcal{D}^c_\pm(\lambda)$ by

$$P^c_\pm(\lambda) = \pm\gamma_{0^\pm}\mathcal{D}^c(\lambda)^{-1}\gamma_0^* G.$$

One can show that $\mathcal{D}^c(0)$ is invertible and $P^c_\pm(0) = \mathcal{C}^c_\pm$. In the following lemma we summarize the properties of $P^c_\pm(\lambda)$.

Lemma 3.1. *For all $\lambda \in \Lambda \cup \{0\}$, the operators $P^c_\pm(\lambda)$ over $C^\infty(Y, S_0)$ are projections satisfying*

$$P^c_+(\lambda) + P^c_-(\lambda) = \mathrm{Id} \tag{3.1}$$

and the images of $P^c_\pm(\lambda)$ over $C^\infty(Y, S_0)$ coincide with the Cauchy data spaces

$$\mathcal{H}^c_\pm(\lambda) = \big\{\gamma_{0^\pm}\phi_\pm \mid \phi_\pm \in C^\infty(N_\pm, S) \cap \mathrm{dom}(\mathcal{D}^c_\pm(\lambda)),\ \mathcal{D}^c_\pm(\lambda)\phi_\pm = 0\big\}.$$

Proof. The proof of this lemma is similar to Seeley [42, Theorem 5] (see also [13]). We shall prove that $P^c_-(\lambda)$ is a projection with image $\mathcal{H}^c_-(\lambda)$; a similar proof works for $P_+^c(\lambda)$. We first show that $P_-^c(\lambda) = \mathrm{Id}$ on $\mathcal{H}_-^c(\lambda)$. Let $\varphi = \gamma_{0-}\phi_-$, where $\phi_- \in C^\infty(N_-, S) \cap \mathrm{dom}(\mathcal{D}_-^c(\lambda))$ and $\mathcal{D}_-^c(\lambda)\phi_- = 0$, and define

$$\phi := \begin{cases} \phi_- & \text{on } N_-, \\ 0 & \text{on } N \setminus N_-. \end{cases}$$

Since $\mathcal{D}_-^c(\lambda)\phi_- = 0$ and $\mathcal{D}^c(\lambda) = G(\partial_u + D_Y) - \lambda$, and the derivative of the Heaviside function is the delta distribution, it follows that

$$\mathcal{D}^c(\lambda)\phi = -\delta_Y \otimes G\varphi = -\gamma_0^* G\varphi,$$

since $\gamma_0^* = \delta_Y \otimes \cdot$ with δ_Y the delta distribution concentrated at $\{0\} \times Y$. Thus, $\phi = -\mathcal{D}^c(\lambda)^{-1}\gamma_0^* G\varphi$, and so

$$P_-^c(\lambda)\varphi := -\gamma_{0-}\big(\mathcal{D}^c(\lambda)^{-1}\gamma_0^* G\varphi\big) = \gamma_{0-}\big(\phi\big) = \varphi.$$

Hence, $P_-^c(\lambda) = \mathrm{Id}$ on $\mathcal{H}_-^c(\lambda)$. We now show that $P_-^c(\lambda)^2 = P_-^c(\lambda)$. Let $\varphi \in C^\infty(Y, S_0)$. Then by definition of $P_-^c(\lambda)$, we have

$$P_-^c(\lambda)\varphi = \gamma_{0-}\phi, \quad \phi = -\big(\mathcal{D}^c(\lambda)^{-1}\gamma_0^* G\varphi\big)\big|_{N_-}.$$

Note that $\phi \in C^\infty(N_-, S) \cap \mathrm{dom}(\mathcal{D}_-^c(\lambda))$ and $\mathcal{D}_-^c(\lambda)\phi = 0$. Thus, $\gamma_{0-}\phi \in \mathcal{H}_-^c(\lambda)$, so as we know that $P_-^c(\lambda) = \mathrm{Id}$ on $\mathcal{H}_-^c(\lambda)$, it follows that

$$P_-^c(\lambda)^2\varphi = P_-^c(\lambda)\big(P_-^c(\lambda)\varphi\big) = P_-^c(\lambda)\big(\gamma_{0-}\phi\big) = \gamma_{0-}\phi = P_-^c(\lambda)\varphi.$$

We now prove that $P_-^c(\lambda) + P_+^c(\lambda) = \mathrm{Id}$. Let $\phi \in C_c^\infty((-1,1) \times Y, S)$ and let $\psi \in C^\infty(Y, S_0)$. Denote the L^2-pairing on Y by $\langle\,,\rangle$ and denote the distributional pairing on N by parentheses. If $\mathcal{K}^c(\lambda) = \mathcal{D}^c(\lambda)^{-1}\gamma_0^* G$, then

$$\begin{aligned} \langle \gamma_0\phi, G\psi\rangle &= (\gamma_0^* G\psi)(\phi) = \big(\mathcal{D}^c(\lambda)\mathcal{K}^c(\lambda)\psi\big)(\phi) \\ &= \big(\mathcal{K}^c(\lambda)\psi\big)\big(\mathcal{D}^c(\lambda)^*\phi\big) = \int_{-1}^{1} \langle \mathcal{D}^c(\lambda)^*\phi, \mathcal{K}^c(\lambda)\psi\rangle\, du. \end{aligned} \tag{3.2}$$

Since the function $\mathcal{K}^c(\lambda)\psi$ is smooth off Y with at most a jump discontinuity at Y and $\phi \in C_c^\infty((-1,1) \times Y, S)$, we can write

$$\int_{-1}^{1} \langle \mathcal{D}^c(\lambda)^*\phi, \mathcal{K}^c(\lambda)\psi\rangle\, du = \lim_{\varepsilon \to 0^+} \int_{|u|>\varepsilon} \langle \mathcal{D}^c(\lambda)^*\phi, \mathcal{K}^c(\lambda)\psi\rangle\, du.$$

Now observe that

$$\begin{aligned}\int_{|u|>\varepsilon} \langle \mathcal{D}^c(\lambda)^*\phi, \mathcal{K}^c(\lambda)\psi\rangle\, du &= \int_{|u|>\varepsilon} \langle \big(G(\partial_u + D_Y) - \overline{\lambda}\big)\phi, \mathcal{K}^c(\lambda)\psi\rangle\, du \\ &= -\int_{|u|>\varepsilon} \langle \partial_u\phi, G\mathcal{K}^c(\lambda)\psi\rangle\, du \\ &\quad + \int_{|u|>\varepsilon} \langle \phi, \big(GD_Y - \lambda\big)\mathcal{K}^c(\lambda)\psi\rangle\, du \\ &= -\int_{|u|>\varepsilon} \partial_u\langle \phi, G\mathcal{K}^c(\lambda)\psi\rangle\, du \\ &\quad + \int_{|u|>\varepsilon} \langle \phi, \mathcal{D}^c(\lambda)\mathcal{K}^c(\lambda)\psi\rangle\, du \\ &= -\langle \gamma_{-\varepsilon}\phi, G\gamma_{-\varepsilon}\mathcal{K}^c(\lambda)\psi\rangle + \langle \gamma_{\varepsilon}\phi, G\gamma_{\varepsilon}\mathcal{K}^c(\lambda)\psi\rangle,\end{aligned} \tag{3.3}$$

where at the last step we used that $\mathcal{D}^c(\lambda)\mathcal{K}^c(\lambda) = 0$ off Y and the fundamental theorem of calculus, recalling that ϕ is supported away from $u = \pm 1$. Taking $\varepsilon \to 0^+$ in (3.3) and equating this with (3.2), and using the definition of $P_-^c(\lambda)$ and $P_+^c(\lambda)$, we conclude that

$$\langle \gamma_0\phi, G\psi\rangle = \langle \gamma_0\phi, GP_-^c(\lambda)\psi\rangle + \langle \gamma_0\phi, GP_+^c(\lambda)\psi\rangle.$$

Since $\phi \in C_c^\infty((-1,1)\times Y, S)$ and $\psi \in C^\infty(Y, S_0)$ were arbitrary, it follows that $\mathrm{Id} = P_-^c(\lambda) + P_+^c(\lambda)$, and our proof is now complete. □

We now compare the Calderón projectors of our original problem to those of our model problem. To this end, we first define a space of parameter-dependent smoothing operators. For any $p \in \mathbb{Z}$, we define $\Psi_\Lambda^{-\infty,p}(Y, S_0)$ as the space of smoothing operators $\mathcal{S}(\lambda)$ over Y depending smoothly on the parameter $\lambda \in \Lambda$ such that as $|\lambda| \to \infty$ in Λ, we have

$$\mathcal{S}(\lambda) \sim \sum_{j=0}^{\infty} |\lambda|^{p-j}\, \mathcal{S}_j(\theta), \tag{3.4}$$

where $\mathcal{S}_j(\theta) \in \Psi^{-\infty}(Y, S_0)$ (the space of smoothing operators on Y) depends smoothly on $\theta := \lambda/|\lambda|$. If $\mathcal{S}(\lambda)$ happens to depend holomorphically on λ, then the asymptotic sum (3.4) can be written with $|\lambda|$ replaced with λ and where $\mathcal{S}_j$ is independent of θ. Note that $\Psi^{-\infty}(Y, S_0) \subset \Psi_\Lambda^{-\infty,0}(Y, S_0)$.

Proposition 3.2. *The differences* $\mathcal{C}_\pm - \mathcal{C}_\pm^c$ *are in* $\Psi^{-\infty}(Y, S_0)$ *and the differences* $P_\pm(\lambda) - P_\pm^c(\lambda)$ *are in* $\Psi_\Lambda^{-\infty,-2}(Y, S_0)$.

Proof. We first prove the statement for the λ-dependent operators. The main idea follows from the observation that we can replace $\mathcal{D}(\lambda)^{-1}$ by a suitable *parametrix*, which involves the operator $\mathcal{D}^c(\lambda)^{-1}$ up to a smoothing operator plus an integral operator whose support is far from $\{0\} \times Y$.

Let $\rho(a, b) : [-1, 1] \to [0, 1]$ be a smooth even function equal to 0 for $-a \leqslant u \leqslant a$ and equal to 1 for $b \leqslant |u|$. We define

$$\phi_1 = 1 - \rho(\tfrac{5}{7}, \tfrac{6}{7}),\ \phi_2 = \rho(\tfrac{1}{7}, \tfrac{2}{7}), \psi_2 = \rho(\tfrac{3}{7}, \tfrac{4}{7}),\ \psi_1 = 1 - \psi_2$$

and then we extend these functions to be functions on M in the obvious way. We now define a parametrix $Q(\lambda)$ for the operator $\mathcal{D}(\lambda)^{-1}$ by

$$Q(\lambda)(x, z) = \phi_1(x)\mathcal{D}^c(\lambda)^{-1}(x, z)\psi_1(z) + \phi_2(x)\mathcal{D}(\lambda)^{-1}(x, z)\psi_2(z). \tag{3.5}$$

Then we have

$$\begin{aligned}\mathcal{D}(\lambda)Q(\lambda)(x, z) \ &= \ \mathrm{Id} + G\,(\partial_u\phi_1)(x)\,\mathcal{D}^c(\lambda)^{-1}(x, z)\psi_1(z)\\ &\quad + G\,(\partial_u\phi_2)(x)\,\mathcal{D}(\lambda)^{-1}(x, z)\psi_2(z).\end{aligned}$$

Hence,

$$\mathcal{D}(\lambda)Q(\lambda) \ = \ \mathrm{Id} + \mathcal{S}(\lambda),$$

where $\mathcal{S}(\lambda)$ is a smoothing operator whose kernel $\mathcal{S}(\lambda)(x, z)$ is equal to 0 if the distance from x to z is smaller than $\frac{1}{7}$. Since the supports of $\partial_u\phi_i$ and ψ_i are disjoint, it follows from work in [16] that $\mathcal{S}(\lambda) \in \Psi_\Lambda^{-\infty,-1}(M, S)$. Thus,

$$\mathcal{D}(\lambda)^{-1} - Q(\lambda) = \mathcal{S}'(\lambda), \tag{3.6}$$

where $\mathcal{S}'(\lambda) = -\mathcal{D}(\lambda)^{-1}\mathcal{S}(\lambda) \in \Psi_\Lambda^{-\infty,-2}(M, S)$. Finally, using the definitions of $P_\pm(\lambda)$ and $P^c_\pm(\lambda)$ and the equalities (3.5) and (3.6), we obtain

$$P_\pm(\lambda) - P^c_\pm(\lambda) = \pm\gamma_{0\pm}(\mathcal{D}(\lambda)^{-1} - \mathcal{D}^c(\lambda)^{-1})\gamma_0^* G = \pm\gamma_{0\pm}\mathcal{S}'(\lambda)\gamma_0^* G.$$

It follows that $P_\pm(\lambda) - P^c_\pm(\lambda) \in \Psi_\Lambda^{-\infty,-2}(Y, S_0)$.

We now prove that $\mathcal{C}_- - \mathcal{C}^c_-$ is in $\Psi_\Lambda^{-\infty,0}(Y, S_0)$, with a similar argument holding for the "+" operators. For this, we use that $\mathcal{C}_-$ is given by the formula (2.2) with $\mathcal{D}(\lambda)^{-1}$ replaced by $\widetilde{\mathcal{D}}^{-1}$ where $\widetilde{\mathcal{D}}$ is the invertible double of $\mathcal{D}_-$ on $M_- \cup (-M_-)$, cf. [4]. Then replacing $\mathcal{D}(\lambda)^{-1}$ and $\mathcal{D}^c(\lambda)^{-1}$ by $\widetilde{\mathcal{D}}^{-1}$ and $\mathcal{D}^c(0)^{-1}$, respectively, in the definition of the parametrix (3.5), and then proceeding as we did before, proves that $\mathcal{C}_- - \mathcal{C}^c_-$ is in $\Psi_\Lambda^{-\infty,0}(Y, S_0)$. $\square$

We next compare the Poisson operators of our original problem to those of the model problem. For $\mathcal{D}^c_\pm(\lambda)$, we impose the boundary conditions defined by the projections $\mathcal{C}^c_\pm$ at $\{0\} \times Y$ and we denote by $\mathcal{D}^c_{\mathcal{C}^c_\pm}(\lambda)$ the resulting operators. We also define $\mathcal{D}^c_\pm(\lambda)^{-1}$ as we did for $\mathcal{D}_\pm(\lambda)^{-1}$. Observe that the operators $\mathcal{D}^c_{\mathcal{C}^c_\pm}(\lambda)^{-1}$ and $\mathcal{D}^c_\pm(\lambda)^{-1}$ can be extended as the zero maps over $L^2(N_\pm, S)^\perp \subset L^2(M, S)$ and we obtain operators acting on $L^2(M, S)$; we use the same notations for these extensions. Just as we did in (2.4) and (2.5) for the operators $\mathcal{D}_\pm(\lambda)$, we can also define operators $S^c_\pm(\lambda)$, $S^c_\pm(\lambda)^{-1}$, $\mathcal{K}^c_\pm(\lambda)$, and $\mathcal{K}^c_{\mathcal{C}^c_\pm}(\lambda)$ for the operator $\mathcal{D}^c_\pm(\lambda)$. In particular, in an obvious way, we can regard the operators $\mathcal{K}^c_\pm(\lambda)$ and $\mathcal{K}^c_{\mathcal{C}^c_\pm}(\lambda)$ as mapping $L^2(Y, S_0)$ into $L^2(M, S)$. Then using formulas (3.5) and (3.6) in Proposition 3.2, it is straightforward to prove

Lemma 3.3. *The differences $S_\pm(\lambda) - S^c_\pm(\lambda)$ and $S_\pm(\lambda)^{-1} - S^c_\pm(\lambda)^{-1}$ are smoothing operators over $L^2(Y, S_0)$, $\mathcal{K}_{\mathcal{C}_\pm}(\lambda) - \mathcal{K}^c_{\mathcal{C}^c_\pm}(\lambda) : L^2(Y, S_0) \to L^2(M_\pm, S)$ has a smoothing Schwartz kernel apart from a jump discontinuity at $\{\pm 1\} \times Y$ in $M_\pm$, and finally, $\gamma_0(\mathcal{D}_\pm(\lambda)^{-1} - \mathcal{D}^c_\pm(\lambda)^{-1}) : L^2(M_\pm, S) \to L^2(Y, S_0)$ has a smoothing Schwartz kernel apart from a jump discontinuity at $\{\pm 1\} \times Y$ in $M_\pm$.*

Note that the jump discontinuity in the kernel of $\mathcal{K}_{\mathcal{C}_\pm}(\lambda) - \mathcal{K}^c_{\mathcal{C}^c_\pm}(\lambda)$ occurs because $\mathcal{K}^c_{\mathcal{C}^c_\pm}(\lambda)$ maps into $L^2(N_\pm, S)$, so $\mathcal{K}^c_{\mathcal{C}^c_\pm}(\lambda)$ is identically zero off $N_\pm$ when considered as a map into $L^2(M_\pm, S)$. A similar remark holds for $\gamma_0(\mathcal{D}_\pm(\lambda)^{-1} - \mathcal{D}^c_\pm(\lambda)^{-1})$.

Finally, we compare the resolvents of our original problem to those of the model problem. First, we note that a similar formula to (2.7) holds:

$$\mathcal{D}^c_{\mathcal{C}^c_\pm}(\lambda)^{-1} = \mathcal{D}^c_\pm(\lambda)^{-1} - \mathcal{K}^c_{\mathcal{C}^c_\pm}(\lambda)\mathcal{C}^c_\pm\gamma_0\mathcal{D}^c_\pm(\lambda)^{-1}. \tag{3.7}$$

Second, we note that the Hilbert spaces $L^2(M, S)$, $L^2(N, S)$ have the decompositions

$$L^2(M, S) = L^2(M_+, S) \oplus L^2(M_-, S), \;\; L^2(N, S) = L^2(N_+, S) \oplus L^2(N_-, S).$$

With respect these decompositions, $\mathcal{D}(\lambda)^{-1}$, $\mathcal{D}^c(\lambda)^{-1}$ have the matrix forms

$$\mathcal{D}(\lambda)^{-1} = \begin{pmatrix} \mathcal{D}_+(\lambda)^{-1} & r_+\mathcal{D}(\lambda)^{-1}e_- \\ r_-\mathcal{D}(\lambda)^{-1}e_+ & \mathcal{D}_-(\lambda)^{-1} \end{pmatrix},$$
$$\mathcal{D}^c(\lambda)^{-1} = \begin{pmatrix} \mathcal{D}^c_+(\lambda)^{-1} & r_+\mathcal{D}^c(\lambda)^{-1}e_- \\ r_-\mathcal{D}^c(\lambda)^{-1}e_+ & \mathcal{D}^c_-(\lambda)^{-1} \end{pmatrix}. \tag{3.8}$$

Here, r_+, r_- are the restriction maps to M_+, M_- and e_+, e_- are the extension maps by zero out of M_+, M_-. To simplify notation, from now on we put

$$\mathcal{D}_\mathcal{C}(\lambda)^{-1} := \mathcal{D}_{\mathcal{C}_+}(\lambda)^{-1} \sqcup \mathcal{D}_{\mathcal{C}_-}(\lambda)^{-1}, \quad \mathcal{D}^c_{\mathcal{C}^c}(\lambda)^{-1} := \mathcal{D}^c_{\mathcal{C}^c_+}(\lambda)^{-1} \sqcup \mathcal{D}^c_{\mathcal{C}^c_-}(\lambda)^{-1}.$$

Proposition 3.4. *The following operator is of trace class*:

$$\mathcal{D}(\lambda)^{-1} - \mathcal{D}_{\mathcal{C}}(\lambda)^{-1} - (\mathcal{D}^c(\lambda)^{-1} - \mathcal{D}^c_{\mathcal{C}^c}(\lambda)^{-1}) \quad over\ L^2(M,S).$$

Proof. By the formulas (2.7) and (3.7) for $\mathcal{D}_{\mathcal{C}_\pm}(\lambda)^{-1}$ and $\mathcal{D}^c_{\mathcal{C}^c_\pm}(\lambda)^{-1}$, we have

$$\begin{aligned}&\mathcal{D}_\pm(\lambda)^{-1} - \mathcal{D}_{\mathcal{C}_\pm}(\lambda)^{-1} - (\mathcal{D}^c_\pm(\lambda)^{-1} - \mathcal{D}^c_{\mathcal{C}^c_\pm}(\lambda)^{-1})\\ &\quad = -\mathcal{K}_{\mathcal{C}_\pm}(\lambda)\mathcal{C}_\pm\gamma_0\mathcal{D}_\pm(\lambda)^{-1} + \mathcal{K}^c_{\mathcal{C}^c_\pm}(\lambda)\mathcal{C}^c_\pm\gamma_0\mathcal{D}^c_\pm(\lambda)^{-1}.\end{aligned}$$

By Proposition 3.2 and Lemma 3.3, it follows that the right-hand side is of trace class over $L^2(M,S)$. Therefore, if $\mathcal{D}_d(\lambda)^{-1}$ and $\mathcal{D}^c_d(\lambda)^{-1}$ are the operators defined by the diagonal terms of the matrices (3.8), then the operator

$$\mathcal{D}_d(\lambda)^{-1} - \mathcal{D}_{\mathcal{C}}(\lambda)^{-1} - (\mathcal{D}^c_d(\lambda)^{-1} - \mathcal{D}^c_{\mathcal{C}^c}(\lambda)^{-1})$$

is of trace class over $L^2(M,S)$. The claim now follows from the fact the difference of $\mathcal{D}(\lambda)^{-1}$ ($\mathcal{D}^c(\lambda)^{-1}$) and $\mathcal{D}_d(\lambda)^{-1}$ ($\mathcal{D}^c_d(\lambda)^{-1}$) is given by off diagonal terms in the decomposition (3.8).

For an alternative proof, one can take an arbitrary $\phi \in L^2(M,S)$ and consider the sections $\mathcal{D}(\lambda)^{-1}\phi$, $\mathcal{D}_{\mathcal{C}}(\lambda)^{-1}\phi$, $\mathcal{D}^c(\lambda)^{-1}\phi$, and $\mathcal{D}^c_{\mathcal{C}^c}(\lambda)^{-1}\phi$, and show that the alternating sum of these sections is smooth over $M_+ \sqcup M_-$ except with jump discontinuities at $\{\pm 1\} \times Y$ (because $\mathcal{D}^c(\lambda)^{-1}\phi$, and $\mathcal{D}^c_{\mathcal{C}^c}(\lambda)^{-1}\phi$ vanish outside of N); this implies that the operator in question is trace class. □

Remark 3.5. Proposition 3.4 is a crucial element in the proof of Theorem 1.1 because the trace $\mathrm{Tr}(\mathcal{D}(\lambda)^{-1} - \mathcal{D}_{\mathcal{C}}(\lambda)^{-1} - (\mathcal{D}^c(\lambda)^{-1} - \mathcal{D}^c_{\mathcal{C}^c}(\lambda)^{-1}))$ is related to the variation of the log of the ratio of the relative ζ-determinants of $\mathcal{D}^2 - \lambda^2, \mathcal{D}^2_{\mathcal{C}} - \lambda^2$ and $(\mathcal{D}^c)^2 - \lambda^2, (\mathcal{D}^c_{\mathcal{C}^c})^2 - \lambda^2$ (see Proposition 8.4 where we set $\lambda = iv$ with $v \in \mathbb{R}$). We also remark that we can regularize the trace of $\mathcal{D}(\lambda)^{-1} - \mathcal{D}_{\mathcal{C}}(\lambda)^{-1}$ by subtracting off other operators instead of the model cylinder operators. However, the reason we choose the finite cylinder operators is because the ζ-determinants of these operators can be computed *exactly* (see Lemma 9.1). This 'comparison with model problems' technique can also be found in [25,27] where we investigate similar gluing problems.

4. Variation of $\log \det_F (K(\lambda)K^c(\lambda)^{-1})$

In this section, we define the key operators of this paper, $K(\lambda)$ and $K^c(\lambda)$, over Y, which are defined through our various Calderón projectors.

Recall from (1.4) that $\mathcal{C}_\pm$ have the matrix forms

$$\mathcal{C}_+ = \frac{1}{2}\begin{pmatrix} \mathrm{Id} & \kappa_+^{-1} \\ \kappa_+ & \mathrm{Id} \end{pmatrix}, \quad \mathcal{C}_- = \frac{1}{2}\begin{pmatrix} \mathrm{Id} & \kappa_-^{-1} \\ \kappa_- & \mathrm{Id} \end{pmatrix} \tag{4.1}$$

with respect to $L^2(Y, S_0) = L^2(Y, S^+) \oplus L^2(Y, S^-)$, where the operators $\kappa_\pm$ are isometries from $L^2(Y, S^+)$ to $L^2(Y, S^-)$. Now using this decomposition we define

$$V = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\kappa_-\kappa_+^{-1} \end{pmatrix} \quad \text{over} \quad L^2(Y, S^+) \oplus L^2(Y, S^-). \tag{4.2}$$

Then V is a unitary operator on $L^2(Y, S_0)$ and

$$V(\mathrm{Id} - \mathcal{C}_+)V^{-1} = \mathcal{C}_-.$$

For $\lambda \in \Lambda \cup \{0\}$, we introduce $K(\lambda)$, $K^c(\lambda)$ acting on $L^2(Y, S_0)$,

$$\begin{aligned} K(\lambda) &= S_+(\lambda)^{-1} + S_-(\lambda)^{-1}\mathcal{C}_- V(\mathrm{Id} - \mathcal{C}_+), \\ K^c(\lambda) &= S^c_+(\lambda)^{-1} + S^c_-(\lambda)^{-1}. \end{aligned}$$

Remark 4.1. To see why these operators are primal to the gluing problem, consider the following formal argument. Let us focus on $K(\lambda)$. First of all, the factor $\mathcal{C}_- V(\mathrm{Id}-\mathcal{C}_+)$ in front of $S_-(\lambda)^{-1}$ allows $K(\lambda)$ to be written as a diagonal matrix with respect to the following direct sums (the second is non-orthogonal) of $L^2(Y, S_0)$:

$$K(\lambda) = \begin{pmatrix} S_+(\lambda)^{-1} & 0 \\ 0 & S_-(\lambda)^{-1}\mathcal{C}_- V(\mathrm{Id} - \mathcal{C}_+) \end{pmatrix} : \begin{matrix} \mathrm{ran}(\mathcal{C}_+) \\ \oplus \\ \mathrm{ran}(\mathrm{Id} - \mathcal{C}_+) \end{matrix} \to \begin{matrix} \mathrm{ran}(P_+(\lambda)) \\ \oplus \\ \mathrm{ran}(P_-(\lambda)) \end{matrix} . \tag{4.3}$$

Second, recalling that $S_+(\lambda) = \mathcal{C}_+ P_+(\lambda) = \mathcal{C}_+\gamma_{0^+}\mathcal{D}(\lambda)^{-1}\gamma_0^* G$, proceeding *formally* (as the following operators are not of trace class), observe that

$$\begin{aligned} \mathrm{Tr}(\partial_\lambda S_+(\lambda)\, S_+(\lambda)^{-1}) &= \mathrm{Tr}(\mathcal{C}_+\gamma_{0^+}\mathcal{D}(\lambda)^{-2}\gamma_0^* G\, S_+(\lambda)^{-1}) \\ &= \mathrm{Tr}\big(\mathcal{C}_+\gamma_{0^+}\mathcal{D}(\lambda)^{-1}\mathcal{D}(\lambda)^{-1}\gamma_0^* G\, S_+(\lambda)^{-1}\big) \\ &= \mathrm{Tr}\big(\mathcal{D}_+(\lambda)^{-1}\gamma_0^* G\, S_+(\lambda)^{-1}\, \mathcal{C}_+\gamma_0\mathcal{D}_+(\lambda)^{-1}\big) \\ &= \mathrm{Tr}\big(\mathcal{K}_{\mathcal{C}_+}(\lambda)\mathcal{C}_+\gamma_0\mathcal{D}_+(\lambda)^{-1}\big) \quad \text{(by (2.5))} \\ &= \mathrm{Tr}\big(\mathcal{D}_+(\lambda)^{-1} - \mathcal{D}_{\mathcal{C}_+}(\lambda)^{-1}\big) \quad \text{(by (2.7))}. \end{aligned}$$

Thus, in view of the diagonal decomposition (4.3), *formally* we have

$$\begin{aligned}\partial_\lambda \log \det_F K(\lambda) &= -\mathrm{Tr}\big(\mathcal{D}_+(\lambda)^{-1} - \mathcal{D}_{\mathcal{C}_+}(\lambda)^{-1}\big) - \mathrm{Tr}\big(\mathcal{D}_-(\lambda)^{-1} - \mathcal{D}_{\mathcal{C}_-}(\lambda)^{-1}\big)\\ &= -\mathrm{Tr}\big(\mathcal{D}(\lambda)^{-1} - \mathcal{D}_{\mathcal{C}}(\lambda)^{-1}\big).\end{aligned}$$

Making a similar formal argument with $K^c(\lambda)$, we *formally* obtain

$$\begin{aligned}&\partial_\lambda \log \det_F (K(\lambda)K^c(\lambda)^{-1})\\ &\quad= -\mathrm{Tr}\big(\mathcal{D}(\lambda)^{-1} - \mathcal{D}_{\mathcal{C}}(\lambda)^{-1} - (\mathcal{D}^c(\lambda)^{-1} - \mathcal{D}^c_{\mathcal{C}^c}(\lambda)^{-1})\big).\end{aligned}$$

In Theorem 4.4 we shall establish this variation formula in a long, but careful, argument. Now in view of Remark 3.5, we can see that $\partial_\lambda \log \det_F (K(\lambda)K^c(\lambda)^{-1})$ is related to the variation of a ratio of relative ζ-determinants.

A basic property of these operators $K(\lambda)$, $K^c(\lambda)$ is that they are holomorphic functions of $\lambda \in \Lambda$. Another fundamental property is

Proposition 4.2. *For $\lambda \in \Lambda$, the operators $K(\lambda)$ and $K^c(\lambda)$ are invertible with inverses given by*

$$\begin{aligned}K(\lambda)^{-1} &= S_+(\lambda) + (\mathrm{Id} - \mathcal{C}_+)V^{-1}\mathcal{C}_- S_-(\lambda),\\ K^c(\lambda)^{-1} &= S^c_+(\lambda) + S^c_-(\lambda);\end{aligned}$$

moreover, $K^c(0) = \mathrm{Id}$, so $K^c(\lambda)$ is invertible even for $\lambda = 0$.

Proof. First, from the identities (2.3) and (2.4) it is easy to see that

$$\begin{aligned}K(\lambda)K(\lambda)^{-1} &= \big(S_+(\lambda)^{-1} + S_-(\lambda)^{-1}\mathcal{C}_- V(\mathrm{Id} - \mathcal{C}_+)\big)\\ &\quad\circ \big(S_+(\lambda) + (\mathrm{Id} - \mathcal{C}_+)V^{-1}\mathcal{C}_- S_-(\lambda)\big)\\ &= S_+(\lambda)^{-1}S_+(\lambda) + S_-(\lambda)^{-1}S_-(\lambda) = P_+(\lambda) + P_-(\lambda) = \mathrm{Id}.\end{aligned}$$

Second, using the equalities $P_+(\lambda)P_-(\lambda) = 0$, $P_-(\lambda)P_+(\lambda) = 0$, which follow from (2.3), we also have

$$\begin{aligned}K(\lambda)^{-1}K(\lambda) &= \big(S_+(\lambda) + (\mathrm{Id} - \mathcal{C}_+)V^{-1}\mathcal{C}_- S_-(\lambda)\big)\\ &\quad\circ \big(S_+(\lambda)^{-1} + S_-(\lambda)^{-1}\mathcal{C}_- V(\mathrm{Id} - \mathcal{C}_+)\big)\\ &= \mathcal{C}_+ + (\mathrm{Id} - \mathcal{C}_+) = \mathrm{Id}.\end{aligned}$$

A similar computation shows that $S^c_+(\lambda)+S^c_-(\lambda)$ is the inverse of $K^c(\lambda)$. Since $K^c(0) = S^c_+(0)^{-1} + S^c_-(0)^{-1} = \mathcal{C}^c_+ + \mathcal{C}^c_- = \mathrm{Id}$, our proof is complete. $\square$

We remark that $K(0)$ in general has a nontrivial kernel (see Section 5). In Eq. (7.6), we will see that $K(\lambda)K^c(\lambda)^{-1}$ has the form $\mathrm{Id} + \mathcal{S}(\lambda)$ for a smoothing operator $\mathcal{S}(\lambda)$. Hence, we can define $\det_F(K(\lambda)K^c(\lambda)^{-1})$, which is holomorphic when $K(\lambda)K^c(\lambda)^{-1}$ is defined. Although $K(\lambda)$ and $K^c(\lambda)$ are defined over $\Lambda \cup \{0\}$, both $K(\lambda)$ and $K^c(\lambda)$ have meromorphic extensions over $\mathbb{C}$ with poles on the real axis. Let us choose, *and henceforth fix*, an open simply connected region of the plane containing all of Λ and an interval on the real axis, and a corresponding logarithm $\log\det_F(K(\lambda)K^c(\lambda)^{-1})$ defined and depending holomorphically for λ in this fixed region. We shall compute the variation of $\log\det_F(K(\lambda)K^c(\lambda)^{-1})$, but before we do, we first need

Lemma 4.3. *The following equalities hold*:

$$\partial_\lambda \mathcal{K}_{\mathcal{C}_\pm}(\lambda) = \mathcal{D}_{\mathcal{C}_\pm}(\lambda)^{-1}\mathcal{K}_{\mathcal{C}_\pm}(\lambda), \quad \partial_\lambda \mathcal{K}^c_{\mathcal{C}^c_\pm}(\lambda) = \mathcal{D}^c_{\mathcal{C}^c_\pm}(\lambda)^{-1}\mathcal{K}^c_{\mathcal{C}^c_\pm}(\lambda).$$

Proof. We prove this lemma for the "+" and "non-c" case, the other cases being similar. We take the derivative of the equalities in (2.6) to get

$$-\mathcal{K}_{\mathcal{C}_+}(\lambda) + \mathcal{D}_+(\lambda)\partial_\lambda \mathcal{K}_{\mathcal{C}_+}(\lambda) = 0, \quad \mathcal{C}_+\gamma_0\partial_\lambda\mathcal{K}_{\mathcal{C}_+}(\lambda) = 0.$$

The second equality means that $\partial_\lambda\mathcal{K}_{\mathcal{C}_+}(\lambda)$ is in the domain of $\mathcal{D}_{\mathcal{C}_+}(\lambda)$, and then the first equality establishes our lemma. $\square$

The following theorem is the key ingredient in the proof of our main result, which shows the importance of the operators $K(\lambda)$, $K^c(\lambda)$ as explained in Remark 4.1.

Theorem 4.4. *For $\lambda \in \Lambda$, the following variation formula holds*:

$$\partial_\lambda \log\det_F(K(\lambda)K^c(\lambda)^{-1}) = -\mathrm{Tr}\big(\mathcal{D}(\lambda)^{-1} - \mathcal{D}_{\mathcal{C}}(\lambda)^{-1} - (\mathcal{D}^c(\lambda)^{-1} - \mathcal{D}^c_{\mathcal{C}^c}(\lambda)^{-1})\big).$$

Proof. Noting that

$$\partial_\lambda \log\det_F(K(\lambda)K^c(\lambda)^{-1}) = \mathrm{Tr}\big(K(\lambda)^{-1}\partial_\lambda K(\lambda) - K^c(\lambda)^{-1}\partial_\lambda K^c(\lambda)\big)$$

and using the formula for $K(\lambda)^{-1}$ in Lemma 4.2, we obtain

$$\begin{aligned}
K(\lambda)^{-1}\partial_\lambda K(\lambda) &= \big(S_+(\lambda) + (\mathrm{Id} - \mathcal{C}_+)V^{-1}\mathcal{C}_- S_-(\lambda)\big) \\
&\quad \circ \partial_\lambda\big(S_+(\lambda)^{-1} + S_-(\lambda)^{-1}\mathcal{C}_- V(\mathrm{Id} - \mathcal{C}_+)\big) \\
&= S_+(\lambda)\partial_\lambda S_+(\lambda)^{-1} + (\mathrm{Id} - \mathcal{C}_+)V^{-1}\mathcal{C}_- S_-(\lambda)\ \partial_\lambda S_-(\lambda)^{-1}\mathcal{C}_- V(\mathrm{Id} - \mathcal{C}_+) \\
&\quad + \text{off diagonal terms},
\end{aligned}$$

where "off diagonal" here means with respect to the decomposition $L^2(Y, S_0) = \operatorname{ran}(\mathcal{C}_+) \oplus \operatorname{ran}(\mathrm{Id} - \mathcal{C}_+)$. By definitions and Lemma 4.3, we have

$$\partial_\lambda S_+(\lambda)^{-1} = \partial_\lambda \gamma_0 \mathcal{K}_{\mathcal{C}_+}(\lambda) = \gamma_0 \partial_\lambda \mathcal{K}_{\mathcal{C}_+}(\lambda) = \gamma_0 \mathcal{D}_{\mathcal{C}_+}(\lambda)^{-1} \mathcal{K}_{\mathcal{C}_+}(\lambda).$$

Using this and the resolvent formula (2.7), we obtain

$$\begin{aligned} S_+(\lambda)\partial_\lambda S_+(\lambda)^{-1} &= S_+(\lambda)\gamma_0\big(\mathcal{D}_+(\lambda)^{-1} - \mathcal{K}_{\mathcal{C}_+}(\lambda)\mathcal{C}_+\gamma_0\mathcal{D}_+(\lambda)^{-1}\big)\mathcal{K}_{\mathcal{C}_+}(\lambda)\mathcal{C}_+ \\ &= S_+(\lambda)\gamma_0\mathcal{D}_+(\lambda)^{-1}\mathcal{K}_{\mathcal{C}_+}(\lambda)\mathcal{C}_+ - \mathcal{C}_+\gamma_0\mathcal{D}_+(\lambda)^{-1}\mathcal{K}_{\mathcal{C}_+}(\lambda)\mathcal{C}_+. \end{aligned}$$

Using a similar formula for $S_-(\lambda)\partial_\lambda S_-(\lambda)^{-1}$, we can write

$$\begin{aligned} &K(\lambda)^{-1}\partial_\lambda K(\lambda) \\ &\quad = \mathcal{C}_+\big(S_+(\lambda)\gamma_0\mathcal{D}_+(\lambda)^{-1}\mathcal{K}_{\mathcal{C}_+}(\lambda) - \mathcal{C}_+\gamma_0\mathcal{D}_+(\lambda)^{-1}\mathcal{K}_{\mathcal{C}_+}(\lambda)\big)\mathcal{C}_+ \\ &\qquad + (\mathrm{Id} - \mathcal{C}_+)V^{-1}\mathcal{C}_-\big(S_-(\lambda)\gamma_0\mathcal{D}_-(\lambda)^{-1}\mathcal{K}_{\mathcal{C}_-}(\lambda) \\ &\qquad - \mathcal{C}_-\gamma_0\mathcal{D}_-(\lambda)^{-1}\mathcal{K}_{\mathcal{C}_-}(\lambda)\big)\mathcal{C}_-V(\mathrm{Id} - \mathcal{C}_+) \\ &\qquad + \text{off diagonal terms.} \end{aligned}$$

A similar formula holds for $K^c(\lambda)$, so

$$\begin{aligned} &\partial_\lambda \log \det_F (K(\lambda)K^c(\lambda)^{-1}) \\ &\quad = \mathrm{Tr}\big(\mathcal{K}_{\mathcal{C}_+}(\lambda)S_+(\lambda)\gamma_0\mathcal{D}_+(\lambda)^{-1} - \mathcal{K}^c_{\mathcal{C}^c_+}(\lambda)S^c_+(\lambda)\gamma_0\mathcal{D}^c_+(\lambda)^{-1}\big) \\ &\qquad - \mathrm{Tr}\big(\mathcal{K}_{\mathcal{C}_+}(\lambda)\mathcal{C}_+\gamma_0\mathcal{D}_+(\lambda)^{-1} - \mathcal{K}^c_{\mathcal{C}^c_+}(\lambda)\mathcal{C}^c_+\gamma_0\mathcal{D}^c_+(\lambda)^{-1}\big) \\ &\qquad + \mathrm{Tr}\big(\mathcal{K}_{\mathcal{C}_-}(\lambda)S_-(\lambda)\gamma_0\mathcal{D}_-(\lambda)^{-1} - \mathcal{K}^c_{\mathcal{C}_-}(\lambda)S^c_-(\lambda)\gamma_0\mathcal{D}^c_-(\lambda)^{-1}\big) \\ &\qquad - \mathrm{Tr}\big(\mathcal{K}_{\mathcal{C}_-}(\lambda)\mathcal{C}_-\gamma_0\mathcal{D}_-(\lambda)^{-1} - \mathcal{K}^c_{\mathcal{C}^c_-}(\lambda)\mathcal{C}^c_-\gamma_0\mathcal{D}^c_-(\lambda)^{-1}\big), \end{aligned}$$

where the operators in parentheses are trace class according to Lemma 3.3. By (2.5), we have $\mathcal{K}_{\mathcal{C}_+}(\lambda) = \mathcal{K}_+(\lambda)S_+(\lambda)^{-1}$ where $\mathcal{K}_+(\lambda) = \mathcal{D}_+(\lambda)^{-1}\gamma_0^* G$, with similar formulas for the "−" and "c" operators, which imply that

$$\begin{aligned} &\partial_\lambda \log \det_F (K(\lambda)K^c(\lambda)^{-1}) \\ &\quad = \mathrm{Tr}\big(\mathcal{K}_+(\lambda)P_+(\lambda)\gamma_0\mathcal{D}_+(\lambda)^{-1} - \mathcal{K}^c_+(\lambda)P^c_+(\lambda)\gamma_0\mathcal{D}^c_+(\lambda)^{-1}\big) \\ &\qquad - \mathrm{Tr}\big(\mathcal{K}_{\mathcal{C}_+}(\lambda)\mathcal{C}_+\gamma_0\mathcal{D}_+(\lambda)^{-1} - \mathcal{K}^c_{\mathcal{C}_+}(\lambda)\mathcal{C}^c_+\gamma_0\mathcal{D}^c_+(\lambda)^{-1}\big) \\ &\qquad + \mathrm{Tr}\big(\mathcal{K}_-(\lambda)P_-(\lambda)\gamma_0\mathcal{D}_-(\lambda)^{-1} - \mathcal{K}^c_-(\lambda)P^c_-(\lambda)\gamma_0\mathcal{D}^c_-(\lambda)^{-1}\big) \\ &\qquad - \mathrm{Tr}\big(\mathcal{K}_{\mathcal{C}_-}(\lambda)\mathcal{C}_-\gamma_0\mathcal{D}_-(\lambda)^{-1} - \mathcal{K}^c_{\mathcal{C}_-}(\lambda)\mathcal{C}^c_-\gamma_0\mathcal{D}^c_-(\lambda)^{-1}\big). \end{aligned}$$

Now we claim that the following sum vanishes:

$$\begin{aligned}&\mathrm{Tr}\big(\mathcal{K}_+(\lambda)P_+(\lambda)\gamma_0\mathcal{D}_+(\lambda)^{-1} - \mathcal{K}^c_+(\lambda)P^c_+(\lambda)\gamma_0\mathcal{D}^c_+(\lambda)^{-1}\big)\\ &\quad + \mathrm{Tr}\big(\mathcal{K}_-(\lambda)P_-(\lambda)\gamma_0\mathcal{D}_-(\lambda)^{-1} - \mathcal{K}^c_-(\lambda)P^c_-(\lambda)\gamma_0\mathcal{D}^c_-(\lambda)^{-1}\big).\end{aligned} \tag{4.4}$$

To see this, we note that as in the proof of Lemma 4.3, we can get

$$\partial_\lambda \mathcal{K}_\pm(\lambda) = \mathcal{D}(\lambda)^{-1}\mathcal{K}_\pm(\lambda)$$

and therefore noting that $P_\pm(\lambda) = \gamma_{0\pm}\mathcal{K}_\pm(\lambda)$, we see that

$$\partial_\lambda P_\pm(\lambda) = \gamma_{0\pm}\mathcal{D}(\lambda)^{-1}\mathcal{K}_\pm(\lambda) = \gamma_0\mathcal{D}_\pm(\lambda)^{-1}\mathcal{K}_\pm(\lambda)$$

with similar equalities holding for the remaining terms in (4.4). Now taking the derivative of $P_\pm(\lambda) = P_\pm(\lambda)^2$, we obtain

$$\partial_\lambda P_\pm(\lambda) = \partial_\lambda P_\pm(\lambda)P_\pm(\lambda) + P_\pm(\lambda)\partial_\lambda P_\pm(\lambda)$$

and similar equalities for the other projections, hence

$$\begin{aligned}&\mathrm{Tr}\big(\mathcal{K}_+(\lambda)P_+(\lambda)\gamma_0\mathcal{D}_+(\lambda)^{-1} - \mathcal{K}^c_+(\lambda)P^c_+(\lambda)\gamma_0\mathcal{D}^c_+(\lambda)^{-1}\big)\\ &\quad + \mathrm{Tr}\big(\mathcal{K}_-(\lambda)P_-(\lambda)\gamma_0\mathcal{D}_-(\lambda)^{-1} - \mathcal{K}^c_-(\lambda)P^c_-(\lambda)\gamma_0\mathcal{D}^c_-(\lambda)^{-1}\big)\\ &\qquad = \mathrm{Tr}\big(\gamma_0\mathcal{D}_+(\lambda)^{-1}\mathcal{K}_+(\lambda)P_+(\lambda) + \gamma_0\mathcal{D}_-(\lambda)^{-1}\mathcal{K}_-(\lambda)P_-(\lambda)\\ &\qquad\quad - \gamma_0\mathcal{D}^c_+(\lambda)^{-1}\mathcal{K}^c_+(\lambda)P^c_+(\lambda) - \gamma_0\mathcal{D}^c_-(\lambda)^{-1}\mathcal{K}^c_-(\lambda)P^c_-(\lambda)\big)\\ &\qquad = \mathrm{Tr}\big(\partial_\lambda P_+(\lambda)P_+(\lambda) + \partial_\lambda P_-(\lambda)P_-(\lambda)\\ &\qquad\quad - \partial_\lambda P^c_+(\lambda)P^c_+(\lambda) - \partial_\lambda P^c_-(\lambda)P^c_-(\lambda)\big)\\ &\qquad = \tfrac{1}{2}\partial_\lambda \mathrm{Tr}\big(P_+(\lambda) + P_-(\lambda) - (P^c_+(\lambda) + P^c_-(\lambda))\big) = 0,\end{aligned}$$

where we used the equalities (2.3), (3.1). Thus,

$$\begin{aligned}&\partial_\lambda \log \det{}_F\,(K(\lambda)K^c(\lambda)^{-1})\\ &\quad = -\mathrm{Tr}\big(\mathcal{K}_{\mathcal{C}_+}(\lambda)\mathcal{C}_+\gamma_0\mathcal{D}_+(\lambda)^{-1} - \mathcal{K}^c_{\mathcal{C}^c_+}(\lambda)\mathcal{C}^c_+\gamma_0\mathcal{D}^c_+(\lambda)^{-1}\big)\\ &\qquad - \mathrm{Tr}\big(\mathcal{K}_{\mathcal{C}_-}(\lambda)\mathcal{C}_-\gamma_0\mathcal{D}_-(\lambda)^{-1} - \mathcal{K}^c_{\mathcal{C}_-}(\lambda)\mathcal{C}^c_-\gamma_0\mathcal{D}^c_-(\lambda)^{-1}\big).\end{aligned}$$

Finally, using the notation as in the proof of Proposition 3.4, we see that the right-hand side is exactly

$$\begin{aligned}
&-\mathrm{Tr}\big(\mathcal{K}_{\mathcal{C}_+}(\lambda)\mathcal{C}_+\gamma_0\mathcal{D}_+(\lambda)^{-1} - \mathcal{K}^c_{\mathcal{C}^c_+}(\lambda)\mathcal{C}^c_+\gamma_0\mathcal{D}^c_+(\lambda)^{-1}\big)\\
&\quad - \mathrm{Tr}\big(\mathcal{K}_{\mathcal{C}_-}(\lambda)\mathcal{C}_-\gamma_0\mathcal{D}_-(\lambda)^{-1} - \mathcal{K}^c_{\mathcal{C}_-}(\lambda)\mathcal{C}^c_-\gamma_0\mathcal{D}^c_-(\lambda)^{-1}\big)\\
&\quad = -\mathrm{Tr}\big(\mathcal{D}_d(\lambda)^{-1} - \mathcal{D}_{\mathcal{C}}(\lambda)^{-1} - (\mathcal{D}^c_d(\lambda)^{-1} - \mathcal{D}^c_{\mathcal{C}^c}(\lambda)^{-1})\big)\\
&\quad = -\mathrm{Tr}\big(\mathcal{D}(\lambda)^{-1} - \mathcal{D}_{\mathcal{C}}(\lambda)^{-1} - (\mathcal{D}^c(\lambda)^{-1} - \mathcal{D}^c_{\mathcal{C}^c}(\lambda)^{-1})\big),
\end{aligned}$$

which completes our proof. □

5. Asymptotics of $\det_F(K(\lambda)K^c(\lambda)^{-1})$ for small λ

In this section, we investigate the asymptotics of $\det_F(K(\lambda)K^c(\lambda)^{-1})$ for small λ, which enable us to extract the contribution of the nontrivial kernel of $\mathcal{D}$ to our main results. We start with the following lemma.

Lemma 5.1. *The following equalities hold*:

$$P_\pm(\lambda) = \pm\lambda^{-1}\sum_{k=1}^{h_M}\gamma_0 U_k \otimes G\gamma_0 U_k + Q_\pm(\lambda),$$

where $h_M = \dim\ker(\mathcal{D})$, $\{U_k\}$ *is an orthonormal basis of* $\ker(\mathcal{D})$, *and* $Q_\pm(\lambda)$ *are pseudodifferential operators over* Y *that are regular at* $\lambda = 0$.

Proof. The usual analytic Fredholm theory implies that

$$(\mathcal{D}-\lambda)^{-1} = -\lambda^{-1}\sum_{k=1}^{h_M} U_k\otimes U_k + R(\lambda),$$

where $R(\lambda)$ is a pseudodifferential operator over M that is regular at $\lambda = 0$. Applying this equality to the definition of $P_\pm(\lambda)$, we obtain

$$P_\pm(\lambda) = \pm\gamma_{0^\pm}(\mathcal{D}-\lambda)^{-1}\gamma_0^* G = \mp\lambda^{-1}\sum_{k=1}^{h_M}\gamma_0 U_k\otimes U_k\gamma_0^* G + Q_\pm(\lambda),$$

where $Q_\pm(\lambda)$ are regular at $\lambda = 0$. Now for $\varphi \in C^\infty(Y, S_0)$, denoting the L^2 pairing on $C^\infty(Y, S_0)$ by $\langle\,,\,\rangle$ and the distributional pairing by parentheses, we have

$$U_k(\gamma_0^* G\varphi) = (\gamma_0 U_k)(G\,\varphi) = \langle G\,\varphi, \gamma_0 U_k\rangle = -\langle\varphi, G\gamma_0 U_k\rangle = -(G\gamma_0 U_k)(\varphi).$$

Thus,

$$\mp\lambda^{-1}\sum_{k=1}^{h_M}\gamma_0 U_k\otimes U_k\gamma_0^* G=\pm\lambda^{-1}\sum_{k=1}^{h_M}\gamma_0 U_k\otimes G\gamma_0 U_k,$$

which completes our proof. □

From now on, we put $P^{\pm}:=\frac{\mathrm{Id}\mp iG}{2}$, which is the projection onto $S^{\pm}$. We next make the following observation.

Lemma 5.2. *When* $W=G\gamma_0(\ker(\mathcal{D}))$, *the following equalities hold*:

$$\operatorname{ran}(\mathcal{C}_-)\cap\operatorname{ran}(\mathcal{C}_+)=\gamma_0(\ker(\mathcal{D})),\quad \dim W=\dim\ker(\mathcal{D}),$$

$$V=-iG\quad on\quad \mathcal{W}:=W\oplus GW.$$

Proof. By definition of the Calderón projectors $\mathcal{C}_{\pm}$, elements of the intersection $\operatorname{ran}(\mathcal{C}_-)\cap\operatorname{ran}(\mathcal{C}_+)$ are exactly the restrictions of elements of $\ker(\mathcal{D})$ to Y. This proves that $\operatorname{ran}(\mathcal{C}_-)\cap\operatorname{ran}(\mathcal{C}_+)=\gamma_0(\ker(\mathcal{D}))$, and since G is an isomorphism, we also conclude that $\dim W=\dim\ker(\mathcal{D})$ by the unique continuation theorem for $\mathcal{D}$. From the expressions (4.1) for $\mathcal{C}_{\pm}$, it follows that $\kappa_-=\kappa_+$ over $W^+:=P^+W=P^+GW$. Thus, from the definition of V in (4.2), over either W or GW we have

$$V=\begin{pmatrix}\mathrm{Id} & 0\\ 0 & -\mathrm{Id}\end{pmatrix}. \tag{5.1}$$

Recalling that $G=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}$ completes our proof. □

Now let us recall that

$$K(\lambda)^{-1}=S_+(\lambda)+(\mathrm{Id}-\mathcal{C}_+)V^{-1}\mathcal{C}_-S_-(\lambda).$$

Hence, by Lemma 5.1, we have

$$K(\lambda)^{-1}=\mathcal{C}_+(\lambda^{-1}\widetilde{\mathcal{L}}+Q_+(\lambda))+(\mathrm{Id}-\mathcal{C}_+)V^{-1}\mathcal{C}_-(-\lambda^{-1}\widetilde{\mathcal{L}}+Q_-(\lambda)),$$

where

$$\widetilde{\mathcal{L}}=\sum_{k=1}^{h_M}\gamma_0 U_k\otimes G\gamma_0 U_k.$$

Therefore, we can rewrite $K(\lambda)^{-1}$ as

$$K(\lambda)^{-1} = \lambda^{-1}(\widetilde{\mathcal{L}} - V^{-1}\widetilde{\mathcal{L}}) + Q(\lambda), \tag{5.2}$$

where $Q(\lambda)$ is a pseudodifferential operator over Y that is regular at $\lambda = 0$. From Lemma 5.2, we recall that $V = -iG$ over the space spanned by $\{\gamma_0 U_k\}$. Hence, if we look at the first term in more detail, we have

$$\begin{aligned}\lambda^{-1}(\widetilde{\mathcal{L}} - V^{-1}\widetilde{\mathcal{L}}) &= \lambda^{-1}\Big(\sum_{k=1}^{h_M} \gamma_0 U_k \otimes G\gamma_0 U_k - \sum_{k=1}^{h_M}(-iG)\gamma_0 U_k \otimes G\gamma_0 U_k\Big)\\ &= \lambda^{-1}\sum_{k=1}^{h_M}(i-G)G\gamma_0 U_k \otimes G\gamma_0 U_k = \lambda^{-1}2i\, P^-G\widetilde{\mathcal{L}}. \end{aligned}\tag{5.3}$$

By (5.2), (5.3), and Lemma 5.2, we obtain

Proposition 5.3. *For $\lambda = iv$ where $v \in \mathbb{R} \setminus \{0\}$, we have*

$$K(iv)^{-1} = v^{-1}\big(2P^-G\widetilde{\mathcal{L}}\big) + Q(v), \tag{5.4}$$

where $Q(v)$ is a pseudodifferential operator on Y that is regular at $v = 0$. In particular, the kernel of $K(0)$ is $W^- := P^-W = P^-GW$.

We now analyze the residue operator $2P^-G\widetilde{\mathcal{L}}$ of the right-hand side of (5.4). Recalling $W^\pm = P^\pm W = P^\pm G\gamma_0(\ker(\mathcal{D}))$, let us observe that $P^\pm G : \gamma_0(\ker(\mathcal{D})) \to W^\pm$ are isomorphisms and we define

$$\widetilde{\mathcal{L}}_\pm := 2P^-G\widetilde{\mathcal{L}}P^\pm : W^\pm \to W^-.$$

Then we have

Lemma 5.4. *The following equality holds*:

$$\widetilde{\mathcal{L}}_\pm = (P^-G)\mathcal{L}(P^\pm G)^{-1} : W^\pm \to W^-. \tag{5.5}$$

Proof. From definition of $\mathcal{L} := \sum_{k=1}^{h_M} \gamma_0 U_k \otimes \gamma_0 U_k$, we have

$$\begin{aligned}\Big((P^-G)\mathcal{L}(P^\pm G)^{-1}\Big)(P^\pm G\gamma_0 U_j) &= (P^-G)\,\mathcal{L}\,\gamma_0 U_j\\ &= \sum_{k=1}^{h_M}\big(P^-G\gamma_0 U_k\big)\,\langle\gamma_0 U_j, \gamma_0 U_k\rangle,\end{aligned}\tag{5.6}$$

where $\langle\ ,\ \rangle$ denotes the L^2-inner product on Y. On the other hand, by definition of $\widetilde{\mathcal{L}}_\pm$, we have

$$\widetilde{\mathcal{L}}_\pm\,(P^\pm G\gamma_0 U_j) = 2\sum_{k=1}^{h_M}(P^- G\gamma_0 U_k)\langle P^\pm G\gamma_0 U_j, P^\pm G\gamma_0 U_k\rangle.$$

Now observe that

$$\begin{aligned}\langle\gamma_0 U_j, \gamma_0 U_k\rangle &= \langle G\gamma_0 U_j, G\gamma_0 U_k\rangle\\ &= \langle P^+ G\gamma_0 U_j, P^+ G\gamma_0 U_k\rangle + \langle P^- G\gamma_0 U_j, P^- G\gamma_0 U_k\rangle\\ &= 2\langle P^\pm G\gamma_0 U_j, P^\pm G\gamma_0 U_k\rangle,\end{aligned}$$

since $\langle P^+ G\gamma_0 U_j, P^+ G\gamma_0 U_k\rangle = \langle P^- G\gamma_0 U_j, P^- G\gamma_0 U_k\rangle$. Hence,

$$\widetilde{\mathcal{L}}_\pm\,(P^\pm G\gamma_0 U_j) = \sum_{k=1}^{h_M}(P^- G\gamma_0 U_k)\,\langle\gamma_0 U_j, \gamma_0 U_k\rangle. \tag{5.7}$$

Comparing (5.6) and (5.7) proves (5.5) and completes our proof. □

Let us observe that $\mathcal{W} = W \oplus GW = W^+ \oplus W^-$, then we have

Proposition 5.5. *With respect to the decomposition $L^2(Y, S_0) = \mathcal{W}\oplus\mathcal{W}^\perp$, the operator $K(0)$ over $L^2(Y, S_0)$ takes the matrix form*

$$K(0) = \begin{pmatrix} A & 0 \\ 0 & P_{\mathcal{W}^\perp} K(0) P_{\mathcal{W}^\perp}\end{pmatrix}$$

and the operator $A : \mathcal{W} \to \mathcal{W}$ is of the form $A = \begin{pmatrix} \mathrm{Id} & 0 \\ -\kappa_0 & 0\end{pmatrix}$ with respect to the decomposition $\mathcal{W} = W^+ \oplus W^-$ where $\kappa_0 := \kappa_+|_{W^+} = \kappa_-|_{W^+}$.

Proof. Using that $K(0) = \mathcal{C}_+ + \mathcal{C}_- V(\mathrm{Id} - \mathcal{C}_+)$ by definition of $K(\lambda)$, and that $\mathcal{C}_\pm G = G(\mathrm{Id} - \mathcal{C}_\pm)$, for $\varphi \in W$ we have

$$\mathcal{C}_\pm\varphi = 0, \quad \mathcal{C}_\pm G\varphi = G\varphi.$$

Using these formulas and the fact that $V = -iG$ over $\mathcal{W} = W \oplus GW$ proved in Lemma 5.2 (see the identity (5.1)), we find that

$$K(0)P^+\varphi = (\mathcal{C}_+ + \mathcal{C}_- V(\mathrm{Id} - \mathcal{C}_+))P^+\varphi = V\varphi = P^+\varphi - P^-\varphi,$$
$$K(0)P^-\varphi = (\mathcal{C}_+ + \mathcal{C}_- V(\mathrm{Id} - \mathcal{C}_+))P^-\varphi = 0. \tag{5.8}$$

These equations show that $K(0) : \mathcal{W} \to \mathcal{W}$ and if $A := K(0)|_{\mathcal{W}}$, then with respect to the decomposition $\mathcal{W} = W^+ \oplus W^-$, we have $A = \begin{pmatrix} \mathrm{Id} & 0 \\ -\kappa_0 & 0 \end{pmatrix}$ since $\varphi = (P^+\varphi, P^-\varphi) = (P^+\varphi, -\kappa_0 P^+\varphi) \in W = G\gamma_0(\ker(\mathcal{D}))$. Thus, our proof is finished once we show that $P_{\mathcal{W}^\perp} K(0) P_{\mathcal{W}} = 0$ and $P_{\mathcal{W}} K(0) P_{\mathcal{W}^\perp} = 0$. That $P_{\mathcal{W}^\perp} K(0) P_{\mathcal{W}} = 0$ follows from the fact that $K(0) : \mathcal{W} \to \mathcal{W}$. To prove that $T := P_{\mathcal{W}} K(0) P_{\mathcal{W}^\perp} = 0$ it suffices to consider adjoints and prove that $T^* = P_{\mathcal{W}^\perp} K(0)^* P_{\mathcal{W}} = 0$. However, the exact same argument shown in (5.8) can be used to show that

$$K(0)^* = \mathcal{C}_+ + (\mathrm{Id} - \mathcal{C}_+)V^{-1}\mathcal{C}_- : \mathcal{W} \to \mathcal{W},$$

which in turn proves that $T^* = 0$. $\square$

From Proposition 5.5 and the fact that $K^c(0) = S^c_+(0)^{-1} + S^c_-(0)^{-1} = \mathcal{C}^c_+ + \mathcal{C}^c_- = \mathrm{Id}$, we have

Proposition 5.6. *For small $\lambda = iv$ near* 0, *with respect to the decomposition $L^2(Y, S_0) = \mathcal{W} \oplus \mathcal{W}^\perp$ we have*

$$K(\lambda)K^c(\lambda)^{-1} = \begin{pmatrix} A(v) & \mathcal{O}(v) \\ \mathcal{O}(v) & P_{\mathcal{W}^\perp} K(0) P_{\mathcal{W}^\perp} + \mathcal{O}(v) \end{pmatrix},$$

and with respect to the decomposition $\mathcal{W} = W^+ \oplus W^-$,

$$A(v) = \begin{pmatrix} \mathrm{Id} + \mathcal{O}(v) & \mathcal{O}(v) \\ -\kappa_0 + \mathcal{O}(v) & \mathcal{O}(v) \end{pmatrix}.$$

We are ready to determine the asymptotics of $\det_F\, (K(iv)K^c(iv)^{-1})$ for real v near 0. It follows from Proposition 5.6 that for real v near 0, we have

$$\det_F\, (K(iv)K^c(iv)^{-1}) = \det\, A(v) \cdot \det_{\mathcal{W}^\perp} K(0) \cdot \big(1 + o(1)\big), \tag{5.9}$$

where $\det_{\mathcal{W}^\perp}$ is the Fredholm determinant over the orthogonal complement of $\mathcal{W}$. In the following lemma we investigate the factor $\det A(v)$ on the right-hand side of (5.9).

Lemma 5.7. *For real* v *near* 0, *we have*

$$\det A(v) = v^{h_M} (\det \mathcal{L})^{-1} \big(1 + o(1)\big).$$

Proof. If $B(v) := A(v)^{-1}$, then it is sufficient to prove that for real v near 0,

$$\det B(v) = v^{-h_M} (\det \mathcal{L}) \big(1 + o(1)\big).$$

Now by Proposition 5.3, an elementary matrix computation shows that $B(v)$ must have the form

$$B(v) = \begin{pmatrix} \mathrm{Id} + b\, \widetilde{\mathcal{L}}_-^{-1} \widetilde{\mathcal{L}}_+ + \mathcal{O}(v) & b + \mathcal{O}(v) \\ v^{-1} \widetilde{\mathcal{L}}_+ + p(v) & v^{-1} \widetilde{\mathcal{L}}_- + q(v) \end{pmatrix} \tag{5.10}$$

with respect to the decomposition $\mathcal{W} = W^+ \oplus W^-$, where $p(v), q(v)$ are regular at $v = 0$ and $b : W^- \to W^+$. We can rewrite this as

$$B(v) = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & v^{-1} \widetilde{\mathcal{L}}_- \end{pmatrix} \begin{pmatrix} \mathrm{Id} + b\, \widetilde{\mathcal{L}}_-^{-1} \widetilde{\mathcal{L}}_+ + \mathcal{O}(v) & b + \mathcal{O}(v) \\ \widetilde{\mathcal{L}}_-^{-1} \widetilde{\mathcal{L}}_+ + v \widetilde{\mathcal{L}}_-^{-1} p(v) & \mathrm{Id} + v \widetilde{\mathcal{L}}_-^{-1} q(v) \end{pmatrix}.$$

Thus,

$$\det B(v) = v^{-h_M} (\det \widetilde{\mathcal{L}}_-) \cdot \det \begin{pmatrix} \mathrm{Id} + b\, \widetilde{\mathcal{L}}_-^{-1} \widetilde{\mathcal{L}}_+ + \mathcal{O}(v) & b + \mathcal{O}(v) \\ \widetilde{\mathcal{L}}_-^{-1} \widetilde{\mathcal{L}}_+ + v \widetilde{\mathcal{L}}_-^{-1} p(v) & \mathrm{Id} + v \widetilde{\mathcal{L}}_-^{-1} q(v) \end{pmatrix}.$$

Since $\det \widetilde{\mathcal{L}}_- = \det \mathcal{L}$ by Lemma 5.4, we just have to prove that

$$\det \begin{pmatrix} \mathrm{Id} + b\, \widetilde{\mathcal{L}}_-^{-1} \widetilde{\mathcal{L}}_+ & b \\ \widetilde{\mathcal{L}}_-^{-1} \widetilde{\mathcal{L}}_+ & \mathrm{Id} \end{pmatrix} = 1.$$

But this just follows from the fact

$$\begin{pmatrix} \mathrm{Id} + b\, \widetilde{\mathcal{L}}_-^{-1} \widetilde{\mathcal{L}}_+ & b \\ \widetilde{\mathcal{L}}_-^{-1} \widetilde{\mathcal{L}}_+ & \mathrm{Id} \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & b \widetilde{\mathcal{L}}_-^{-1} \\ 0 & \widetilde{\mathcal{L}}_-^{-1} \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ \widetilde{\mathcal{L}}_+ & \widetilde{\mathcal{L}}_- \end{pmatrix},$$

which implies that

$$\det \begin{pmatrix} \mathrm{Id} + b\, \widetilde{\mathcal{L}}_-^{-1} \widetilde{\mathcal{L}}_+ & b \\ \widetilde{\mathcal{L}}_-^{-1} \widetilde{\mathcal{L}}_+ & \mathrm{Id} \end{pmatrix} = (\det \widetilde{\mathcal{L}}_-^{-1})(\det \widetilde{\mathcal{L}}_-) = 1.$$

This completes the proof. □

For the second factor on the right-hand side of (5.9), we have

Lemma 5.8. *The following equality holds*:

$$\det_{\mathcal{W}^\perp} K(0) = \det_F \Big(\frac{\mathrm{Id} + \widehat{U}}{2}\Big).$$

Proof. Let us consider the equality

$$K(0) = \mathcal{C}_+ + \mathcal{C}_- V(\mathrm{Id} - \mathcal{C}_+) = \begin{pmatrix} \mathcal{C}_+ & \mathcal{C}_+\mathcal{C}_- V(\mathrm{Id} - \mathcal{C}_+) \\ 0 & (\mathrm{Id} - \mathcal{C}_+)\mathcal{C}_- V(\mathrm{Id} - \mathcal{C}_+) \end{pmatrix},$$

which is written with respect to $L^2(Y, S_0) = \mathrm{ran}(\mathcal{C}_+) \oplus \mathrm{ran}(\mathrm{Id} - \mathcal{C}_+)$. Since $W = G(\mathrm{ran}(\mathcal{C}_-) \cap \mathrm{ran}(\mathcal{C}_+)) \subset \mathrm{ran}(\mathrm{Id} - \mathcal{C}_+)$ and $GW \subset \mathrm{ran}(\mathcal{C}_+)$, we see that

$$\det_{\mathcal{W}^\perp} K(0) = \det_{W^\perp} (\mathrm{Id} - \mathcal{C}_+)\mathcal{C}_- V(\mathrm{Id} - \mathcal{C}_+), \tag{5.11}$$

where $\det_{W^\perp}$ is the Fredholm determinant over the orthogonal complement of W within $\mathrm{ran}(\mathrm{Id} - \mathcal{C}_+)$. Then for $\varphi = (x, -\kappa_+ x) \in \mathrm{ran}(\mathrm{Id} - \mathcal{C}_+)$ written as a column vector, using the formulas (4.1) and (4.2) for $\mathcal{C}_\pm$ and V, we have

$$\begin{aligned} &(\mathrm{Id} - \mathcal{C}_+)\mathcal{C}_- V(\mathrm{Id} - \mathcal{C}_+)\varphi \\ &= \frac{1}{2}\begin{pmatrix} \mathrm{Id} & -\kappa_+^{-1} \\ -\kappa_+ & \mathrm{Id} \end{pmatrix} \frac{1}{2}\begin{pmatrix} \mathrm{Id} & \kappa_-^{-1} \\ \kappa_- & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\kappa_-\kappa_+^{-1} \end{pmatrix} \begin{pmatrix} x \\ -\kappa_+ x \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \mathrm{Id} - \kappa_+^{-1}\kappa_- & 0 \\ 0 & \mathrm{Id} - \kappa_-\kappa_+^{-1} \end{pmatrix} \begin{pmatrix} x \\ -\kappa_+ x \end{pmatrix}. \end{aligned}$$

In other words,

$$(\mathrm{Id} - \mathcal{C}_+)\mathcal{C}_- V(\mathrm{Id} - \mathcal{C}_+)\varphi = \Big(\kappa_+^{-1}\frac{\mathrm{Id} - \kappa_-\kappa_+^{-1}}{2}\kappa_+ x, \frac{\mathrm{Id} - \kappa_-\kappa_+^{-1}}{2}(-\kappa_+)x\Big). \tag{5.12}$$

Recalling that $U := -\kappa_-\kappa_+^{-1}$, we observe that for $\psi \in L^2(Y, S^-)$,

$$-U\psi = \kappa_-\kappa_+^{-1}\psi = \psi \quad \text{if and only if} \quad \psi \in W^- = P^- W = P^- G W;$$

that is, the (-1)-eigenspace of U is exactly W^-. Thus, if we define $\widehat{U}$ as the restriction of U to the orthogonal complement of its (-1)-eigenspace, then $\mathrm{Id} + \widehat{U}$ is invertible on its domain. By (5.11) and (5.12), we obtain

$$\det{}_{\mathcal{W}^\perp} K(0) = \det{}_F \left(\frac{\mathrm{Id} + \widehat{U}}{2}\right). \qquad \square$$

Combining (5.9) and Lemmas 5.7 and 5.8, we get

Theorem 5.9. *For real* v *near* 0, *we have*

$$\det{}_F\,(K(iv)K^c(iv)^{-1}) = v^{h_M}\,(\det\,\mathcal{L})^{-1}\det{}_F\left(\frac{\mathrm{Id} + \widehat{U}}{2}\right)\big(1 + o(1)\big).$$

6. Limits of $\det_F\,(K(\lambda)K^c(\lambda)^{-1})$ as $\Im\lambda \to \pm\infty$

In this section, we investigate the limits of $P_\pm(\lambda)$, $S_\pm(\lambda)$, $K(\lambda)$, and the corresponding operators on the model cylinder, as $\Im\lambda \to \pm\infty$ within Λ. We begin with the Calderón projectors.

Theorem 6.1. *For* $\lambda \in \Lambda$, *we have*

$$\lim_{\Im\lambda\to\infty} P_\pm(\lambda) = \lim_{\Im\lambda\to\infty} P^c_\pm(\lambda) = P^\mp$$

and

$$\lim_{\Im\lambda\to-\infty} P_\pm(\lambda) = \lim_{\Im\lambda\to-\infty} P^c_\pm(\lambda) = P^\pm,$$

where $P^\pm = \frac{\mathrm{Id}\mp iG}{2}$, *the projections onto* $L^2(Y, S^\pm)$.

Proof. Since the proofs of each limit are similar, we shall only prove the limit of $P_-(\lambda)$ as $\Im\lambda \to \infty$. Throughout this proof the parameter λ is always restricted to Λ with $\Im\lambda > 0$. Using the fact that

$$\mathcal{D}(\lambda)^{-1} = (\mathcal{D} - \lambda)^{-1} = (\mathcal{D} + \lambda)(\mathcal{D}^2 - \lambda^2)^{-1},$$

we can write $P_-(\lambda)$ as

$$P_-(\lambda) = -\gamma_{0-}(\mathcal{D} + \lambda)(\mathcal{D}^2 - \lambda^2)^{-1}\gamma_0^* G.$$

Since $(\mathcal{D} - \lambda)^{-1}$ is smoothing away from the diagonal and vanishes to infinite order as $|\Im\lambda| \to \infty$ there [16], it suffices to work in a coordinate patch $[-1, 1]_u \times \mathbb{R}^{n-1}$ near the cross section Y, where $\mathbb{R}^{n-1}$ is a coordinate patch on the cross section with

n the dimension of the manifold M. For a compactly supported section $\psi(u, y)$ in this coordinate patch, we can write

$$(\mathcal{D}+\lambda)(\mathcal{D}^2-\lambda^2)^{-1}\psi = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{iu\xi+iy\cdot\eta} d(u, y, \xi, \eta; \lambda)\, \hat{\psi}(\xi, \eta)\, d\xi\, d\eta,$$

where $d(u, y, \xi, \eta; \lambda)$ is the (complete) symbol of $(\mathcal{D}+\lambda)(\mathcal{D}^2-\lambda^2)^{-1}$ and with $\hat{\psi}$ denoting the Fourier transform of ψ. Thus, for a compactly supported section φ on the cross section coordinate patch $\mathbb{R}^{n-1}$, we have

$$\begin{aligned} P_-(\lambda)\varphi &= -\gamma_{0^-}(\mathcal{D}+\lambda)(\mathcal{D}^2-\lambda^2)^{-1}(\gamma_0^* G\varphi) \\ &= -\frac{1}{(2\pi)^{n-1}}\int_{\mathbb{R}^{n-1}} e^{iy\cdot\eta}\, c(y, \eta; \lambda)\, G\hat{\varphi}(\eta)\, d\eta, \end{aligned} \tag{6.1}$$

where

$$c(y, \eta; \lambda) = \lim_{u\to 0^-} \frac{1}{2\pi}\int_{\mathbb{R}} e^{iu\xi}\, d(u, y, \xi, \eta; \lambda)\, d\xi.$$

To determine the symbol $c(y, \eta; \lambda)$ we proceed as follows. First, in view of the decomposition $\mathcal{D} = G(\partial_u + D_Y)$ near the dividing hypersurface Y, the principal symbol of $\mathcal{D}$ is $G\big(i\xi + b(y, \eta)\big)$, where $b(y, \eta)$ is the principal symbol of D_Y. Second, the (complete) symbol of $(\mathcal{D}^2-\lambda^2)^{-1}$ satisfies, cf. [16],

$$\frac{1}{\xi^2+|\eta|^2-\lambda^2} + \mathcal{O}\big((|\xi|+|\eta|+|\lambda|)^{-3}\big),$$

where $|\eta|^2$ is the (Riemannian metric on Y) square of the covector η and the second term is a rational symbol in ξ, η, and λ that is $\mathcal{O}\big((|\xi|+|\eta|+|\lambda|)^{-3}\big)$. It follows that the symbol $d(u, y, \xi, \eta; \lambda)$, which is just the symbol of the operator $(\mathcal{D}+\lambda)(\mathcal{D}^2-\lambda^2)^{-1}$, is given by

$$d(u, y, \xi, \eta; \lambda) = \frac{G\big(i\xi + b(y, \eta)\big)+\lambda}{\xi^2+|\eta|^2-\lambda^2} + \mathcal{O}\big((|\xi|+|\eta|+|\lambda|)^{-2}\big).$$

Thus,

$$\begin{aligned} \frac{1}{2\pi}\int_{\mathbb{R}} e^{iu\xi}\, d(u, y, \xi, \eta; \lambda)\, d\xi &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{iu\xi}\, \frac{G\, i\xi}{\xi^2+|\eta|^2-\lambda^2}\, d\xi \\ &\quad + \frac{1}{2\pi}\int_{\mathbb{R}} e^{iu\xi}\, \frac{Gb(y, \eta)+\lambda}{\xi^2+|\eta|^2-\lambda^2}\, d\xi \\ &\quad + \frac{1}{2\pi}\int_{\mathbb{R}} e^{iu\xi}\, \mathcal{O}\big((|\xi|+|\eta|+|\lambda|)^{-2}\big)\, d\xi. \end{aligned} \tag{6.2}$$

We can evaluate the first term on the right-hand side in (6.2) via the usual technique of contour integration by writing

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{iu\xi}\,\frac{G\,i\xi}{\xi^2+|\eta|^2-\lambda^2}\,d\xi = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iu\xi}\,\frac{G\,i\xi}{\big(\xi+i\sqrt{|\eta|^2-\lambda^2}\big)\big(\xi-i\sqrt{|\eta|^2-\lambda^2}\big)}\,d\xi.$$

Here, for any $\eta\in\mathbb{R}^{n-1}$ and $\lambda\in\Lambda$ with $\Im\lambda>0$, $|\eta|^2-\lambda^2$ is never on the negative real axis, so we can define $\sqrt{|\eta|^2-\lambda^2}$ by taking $-\pi<\arg(|\eta|^2-\lambda^2)<\pi$. Now, shift the contour $\mathbb{R}=\{\Im(\xi)=0\}$ down to $\{\Im(\xi)=-\infty\}$ where at this last contour the integral is zero since $e^{iu\xi}$ will decay exponentially as $\Im(\xi)\to-\infty$ (here we use the so-called *Jordan's inequality* recalling that $u<0$ in M_-). Hence by Cauchy's theorem, the above integral is given in terms of the residue at $\xi=-i\sqrt{|\eta|^2-\lambda^2}$:

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{iu\xi}\,\frac{G\,i\xi}{\big(\xi+i\sqrt{|\eta|^2-\lambda^2}\big)\big(\xi-i\sqrt{|\eta|^2-\lambda^2}\big)}\,d\xi = -i\cdot e^{u\sqrt{|\eta|^2-\lambda^2}}\,\frac{G\,i\big(-i\sqrt{|\eta|^2-\lambda^2}\big)}{-2i\sqrt{|\eta|^2-\lambda^2}} = e^{u\sqrt{|\eta|^2-\lambda^2}}\,\frac{G}{2}.$$

Therefore,

$$\lim_{u\to 0^-}\frac{1}{2\pi}\int_{\mathbb{R}} e^{iu\xi}\,\frac{G\,i\xi}{\xi^2+|\eta|^2-\lambda^2}\,d\xi=\frac{1}{2}G.$$

For the second term on the right-hand side of the equality in (6.2), we make the change of variables $\xi\mapsto\xi\sqrt{|\eta|^2-\lambda^2}$, to get

$$\frac{1}{2\pi}\int_{\mathbb{R}}\frac{1}{\xi^2+|\eta|^2-\lambda^2}\,d\xi=\frac{1}{2\pi}\frac{\sqrt{|\eta|^2-\lambda^2}}{|\eta|^2-\lambda^2}\int_{\mathbb{R}}\frac{1}{\xi^2+1}\,d\xi=\frac{1}{2\sqrt{|\eta|^2-\lambda^2}}.$$

Thus,

$$\lim_{u\to 0^-}\frac{1}{2\pi}\int_{\mathbb{R}} e^{iu\xi}\,\frac{G\,b(y,\eta)+\lambda}{\xi^2+|\eta|^2-\lambda^2}\,d\xi = \frac{1}{2\pi}\int_{\mathbb{R}}\frac{G\,b(y,\eta)+\lambda}{\xi^2+|\eta|^2-\lambda^2}\,d\xi = \frac{G\,b(y,\eta)}{2\sqrt{|\eta|^2-\lambda^2}}+\frac{\lambda}{2\sqrt{|\eta|^2-\lambda^2}}.$$

One can analyze the last term in (6.2) and prove it is $\mathcal{O}\big((|\xi|+|\eta|+|\lambda|)^{-1}\big)$. This shows that

$$\begin{aligned} c(y,\eta;\lambda) &= \lim_{u\to 0^-} \frac{1}{2\pi}\int_{\mathbb{R}} e^{iu\xi}\, d(u,y,\xi,\eta;\lambda)\, d\xi \\ &= \frac{1}{2}G + \frac{G\, b(y,\eta)}{2\sqrt{|\eta|^2-\lambda^2}} + \frac{\lambda}{2\sqrt{|\eta|^2-\lambda^2}} + \mathcal{O}\big((|\xi|+|\eta|+|\lambda|)^{-1}\big) \\ &= \frac{1}{2}\left[G + \frac{\lambda}{\sqrt{|\eta|^2-\lambda^2}} \right] + \mathcal{O}(\lambda^{-1}). \end{aligned} \tag{6.3}$$

Now back to (6.1), we see that

$$\begin{aligned} P_-(\lambda)\varphi &= -\frac{1}{(2\pi)^{n-1}}\int_{\mathbb{R}^{n-1}} e^{iy\cdot\eta}\, c(y,\eta;\lambda)\, G\hat{\varphi}(\eta)\, d\eta \\ &= -\frac{1}{(2\pi)^{n-1}}\int_{\mathbb{R}^{n-1}} e^{iy\cdot\eta}\, \frac{1}{2}\left[G + \frac{\lambda}{\sqrt{|\eta|^2-\lambda^2}} \right] G\hat{\varphi}(\eta)\, d\eta \\ &\quad + \mathcal{O}(\lambda^{-1}). \end{aligned} \tag{6.4}$$

Finally, taking $\Im\lambda \to \infty$, obtain

$$\begin{aligned} \lim_{\Im\lambda\to\infty} P_-(\lambda)\varphi &= -\frac{1}{(2\pi)^{n-1}}\int_{\mathbb{R}^{n-1}} e^{iy\cdot\eta}\, \frac{1}{2}\,[G+i]\; G\hat{\varphi}(\eta)\, d\eta \\ &= -\frac{1}{2}[G+i]\cdot G\varphi = \frac{\mathrm{Id}-iG}{2}\varphi. \qquad \square \end{aligned}$$

Corollary 6.2. *For $\lambda \in \Lambda$, we have*

$$\lim_{\Im\lambda\to\pm\infty} S_\pm(\lambda)^{-1} = \begin{pmatrix} 0 & 0 \\ \kappa_\pm & \mathrm{Id} \end{pmatrix}, \quad \lim_{\Im\lambda\to\pm\infty} S^c_\pm(\lambda)^{-1} = \begin{pmatrix} 0 & 0 \\ \kappa^c_\pm & \mathrm{Id} \end{pmatrix},$$
$$\lim_{\Im\lambda\to\mp\infty} S_\pm(\lambda)^{-1} = \begin{pmatrix} \mathrm{Id} & \kappa_\pm^{-1} \\ 0 & 0 \end{pmatrix}, \quad \lim_{\Im\lambda\to\mp\infty} S^c_\pm(\lambda)^{-1} = \begin{pmatrix} \mathrm{Id} & (\kappa^c_\pm)^{-1} \\ 0 & 0 \end{pmatrix},$$

where the matrices are written with respect to $L^2(Y,S_0) = L^2(Y,S^+) \oplus L^2(Y,S^-)$.

Proof. We prove this lemma only for $S_+(\lambda)^{-1}$ when $\Im\lambda \to \infty$ since the other cases are proved similarly. By Theorem 6.1, as $\Im\lambda \to \infty$, $P_+(\lambda)$ and consequently $P_+^{\mathrm{o}}(\lambda)$ approach the projection P^-. Hence, as $\Im\lambda \to \infty$,

$$\begin{aligned} &\mathcal{C}_+ P_+^{\mathrm{o}}(\lambda) + (\mathrm{Id} - \mathcal{C}_+)(\mathrm{Id} - P_+^{\mathrm{o}}(\lambda)) \\ &\to \frac{1}{2}\begin{pmatrix} \mathrm{Id} & \kappa_+^{-1} \\ \kappa_+ & \mathrm{Id} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \mathrm{Id} & -\kappa_+^{-1} \\ -\kappa_+ & \mathrm{Id} \end{pmatrix}\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 0 & \kappa_+^{-1} \\ 0 & \mathrm{Id} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \mathrm{Id} & 0 \\ -\kappa_+ & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathrm{Id} & \kappa_+^{-1} \\ -\kappa_+ & \mathrm{Id} \end{pmatrix}. \end{aligned}$$

Finding the inverse of the last matrix, we obtain

$$[\mathcal{C}_+ P_+^{\mathrm{o}}(\lambda) + (\mathrm{Id} - \mathcal{C}_+)(\mathrm{Id} - P_+^{\mathrm{o}}(\lambda))\,]^{-1} \to \begin{pmatrix} \mathrm{Id} & -\kappa_+^{-1} \\ \kappa_+ & \mathrm{Id} \end{pmatrix}.$$

Therefore, as $\Im\lambda \to \infty$,

$$\begin{aligned} S_+(\lambda)^{-1} &= P_+^{\mathrm{o}}(\lambda)[\mathcal{C}_+ P_+^{\mathrm{o}}(\lambda) + (\mathrm{Id} - \mathcal{C}_+)(\mathrm{Id} - P_+^{\mathrm{o}}(\lambda))\,]^{-1}\mathcal{C}_+ \\ &\to \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id} \end{pmatrix}\begin{pmatrix} \mathrm{Id} & -\kappa_+^{-1} \\ \kappa_+ & \mathrm{Id} \end{pmatrix}\frac{1}{2}\begin{pmatrix} \mathrm{Id} & \kappa_+^{-1} \\ \kappa_+ & \mathrm{Id} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ \kappa_+ & \mathrm{Id} \end{pmatrix}. \qquad \square \end{aligned}$$

Corollary 6.3. *For $\lambda \in \Lambda$, we have*

$$\lim_{\Im\lambda\to\infty} K(\lambda) = \begin{pmatrix} \mathrm{Id} & -\kappa_+^{-1} \\ \kappa_+ & \mathrm{Id} \end{pmatrix}, \quad \lim_{\Im\lambda\to-\infty} K(\lambda) = \begin{pmatrix} \mathrm{Id} & \kappa_+^{-1} \\ \kappa_- & -\kappa_-\kappa_+^{-1} \end{pmatrix},$$

$$\lim_{\Im\lambda\to\infty} K^c(\lambda) = \begin{pmatrix} \mathrm{Id} & -(\kappa_+^c)^{-1} \\ \kappa_+^c & \mathrm{Id} \end{pmatrix}, \quad \lim_{\Im\lambda\to-\infty} K^c(\lambda) = \begin{pmatrix} \mathrm{Id} & (\kappa_+^c)^{-1} \\ -\kappa_+^c & \mathrm{Id} \end{pmatrix}.$$

Proof. Let us consider the first case. By Corollary 6.2, as $\Im\lambda \to \infty$,

$$K(\lambda) = S_+(\lambda)^{-1} + S_-(\lambda)^{-1}\mathcal{C}_- V(\mathrm{Id} - \mathcal{C}_+)$$

$$\to \begin{pmatrix} 0 & 0 \\ \kappa_+ & \mathrm{Id} \end{pmatrix} + \begin{pmatrix} \mathrm{Id} & \kappa_-^{-1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\kappa_-\kappa_+^{-1} \end{pmatrix} \frac{1}{2} \begin{pmatrix} \mathrm{Id} & -\kappa_+^{-1} \\ -\kappa_+ & \mathrm{Id} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ \kappa_+ & \mathrm{Id} \end{pmatrix} + \begin{pmatrix} \mathrm{Id} & -\kappa_+^{-1} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & -\kappa_+^{-1} \\ \kappa_+ & \mathrm{Id} \end{pmatrix}.$$

The limit $\lim_{\Im\lambda\to-\infty} K(\lambda)$ can be computed using a similar argument. For the remaining cases, we can proceed in the same way and use the relation $\kappa_+^c = -\kappa_-^c$ to get the claimed equalities. $\square$

We are now ready to find the limits of $\det_F (K(\lambda)K^c(\lambda)^{-1})$.

Theorem 6.4. *For $\lambda \in \Lambda$, we have*

$$\lim_{\Im\lambda\to\infty} \det_F (K(\lambda)K^c(\lambda)^{-1}) = 1,$$

$$\lim_{\Im\lambda\to-\infty} \det_F (K(\lambda)K^c(\lambda)^{-1}) = \det_F U.$$

Proof. We only prove the second claim since the first one can be proved in the same way. By Corollary 6.3, we have

$$\lim_{\Im\lambda\to-\infty} \det_F (K(\lambda)K^c(\lambda)^{-1}) = \det_F \begin{pmatrix} \mathrm{Id} & \kappa_+^{-1} \\ \kappa_- & -\kappa_-\kappa_+^{-1} \end{pmatrix} \begin{pmatrix} \mathrm{Id} & (\kappa_+^c)^{-1} \\ -\kappa_+^c & \mathrm{Id} \end{pmatrix}^{-1}.$$

Define

$$A = \begin{pmatrix} 0 & \kappa_+^{-1} \\ -\kappa_+ & 0 \end{pmatrix}, \quad A^c = \begin{pmatrix} 0 & (\kappa_+^c)^{-1} \\ -\kappa_+^c & 0 \end{pmatrix}.$$

Then both A and A^c are unitary operators differing by a smoothing operator such that $A^2 = -\mathrm{Id}$ and $(A^c)^2 = -\mathrm{Id}$. Moreover, for any $t \in \mathbb{R}$, the operator $A(t) := (\mathrm{Id} + tA)(\mathrm{Id} + tA^c)^{-1}$ is of the form $\mathrm{Id} + \text{smoothing}$, and

$$\det_F \begin{pmatrix} \mathrm{Id} & \kappa_+^{-1} \\ \kappa_- & -\kappa_-\kappa_+^{-1} \end{pmatrix} \begin{pmatrix} \mathrm{Id} & (\kappa_+^c)^{-1} \\ -\kappa_+^c & \mathrm{Id} \end{pmatrix}^{-1}$$
$$= \det_F \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\kappa_-\kappa_+^{-1} \end{pmatrix} \begin{pmatrix} \mathrm{Id} & \kappa_+^{-1} \\ -\kappa_+ & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathrm{Id} & (\kappa_+^c)^{-1} \\ -\kappa_+^c & \mathrm{Id} \end{pmatrix}^{-1}$$
$$= \det_F (-\kappa_-\kappa_+^{-1}) \cdot \det_F A(1).$$

Thus, it remains to show that $\det_F\ A(1) = 1$. To see this, observe that $f(t) := \det_F\ A(t)$ is a smooth function of t and $f(0) = \det_F\ \mathrm{Id} = 1$. We claim that $f' = 0$. This shows that f is constant and therefore completes our proof: $\det_F A(1) = f(1) = f(0) = 1$. To prove that $f' = 0$, we make a short computation to obtain

$$f'(t) = \mathrm{Tr}\big(A'(t)\,A(t)^{-1}\big) = \mathrm{Tr}\big((\mathrm{Id} + tA)^{-1}A - (\mathrm{Id} + tA^c)^{-1}A^c\big).$$

Now using that $A^2 = -\mathrm{Id}$ and $(A^c)^2 = -\mathrm{Id}$, it is easy to verify that

$$(\mathrm{Id} + tA)^{-1} = \frac{1}{1+t^2} - \frac{t}{1+t^2}A, \quad (\mathrm{Id} + tA^c)^{-1} = \frac{1}{1+t^2} - \frac{t}{1+t^2}A^c.$$

Replacing these formulas into the formula for $f'(t)$, we get

$$f'(t) = \mathrm{Tr}\left(\frac{A - A^c}{1+t^2}\right).$$

Since $A - A^c$ is off diagonal, this shows that $f' = 0$. □

7. Asymptotics of $\det_F\,(K(\lambda)K^c(\lambda)^{-1})$ for large λ

We begin by briefly reviewing a class of parameter-dependent symbols in [24] that are related to Grubb and Seeley's weakly polyhomogeneous symbols [16,13]. Recall that $\Lambda \subset \mathbb{C}$ is fixed as in (2.1).

For $\mu \in \mathbb{R}$ and $p \in \mathbb{Z}$, we define $S^{\mu,p}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \Lambda)$ as the space of functions $a \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \overline{\Lambda})$, where $\overline{\Lambda} = \Lambda \cup \{0\}$, such that for all multi-indices α, β, γ, and for all $(y, \eta, \lambda) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \overline{\Lambda}$, we have

$$|\partial_y^\gamma \partial_\eta^\alpha \partial_\lambda^\beta a(y, \eta; \lambda)| \leqslant C_{\alpha\beta\gamma}(1 + |\eta|)^{\mu-p-|\alpha|}(1 + |\eta| + |\lambda|)^{p-|\beta|}.$$

The space $S_r^{\mu,p}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \Lambda)$ consists of parameter-dependent symbols $a \in S^{\mu,p}$ $(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \Lambda)$ such that if we set $\lambda = 1/z$ and put

$$\tilde{a}(y, \eta; z) := z^p a(y, \eta; 1/z),$$

then $\tilde{a}(y, \eta; z)$ is smooth at $z = 0$, and

$$|\partial_y^\gamma \partial_\eta^\alpha \partial_z^\beta \tilde{a}(y, \eta; z)| \leqslant C_{\alpha\beta}(1 + |\eta|)^{\mu-p-|\alpha|+|\beta|}(1 + |z||\eta|)^{p-|\beta|} \tag{7.1}$$

uniformly for $|z| \leqslant 1$. Further, let $S^{\mu,p}_{r,\mathrm{cl}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \Lambda)$ be the space of elements $a \in S^{\mu,p}_r(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \Lambda)$ that, for every $N \in \mathbb{N}$, admit a decomposition

$$a(y,\eta;\lambda) = \sum_{j=0}^{N-1} \chi(\eta) a_{\mu-j}(y,\eta;\lambda) + r_N(y,\eta;\lambda), \tag{7.2}$$

where $r_N \in S^{\mu-N,p}_r(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \Lambda)$, $\chi \in C^\infty(\mathbb{R}^{n-1})$ with $\chi(\eta) = 0$ for $|\eta| \leqslant \frac{1}{2}$ and $\chi(\eta) = 1$ for $|\eta| \geqslant 1$, and where each $a_{\mu-j}(y,\eta;\lambda)$ satisfies:

(I) $a_{\mu-j}(y,\eta;\lambda)$ is homogeneous of degree $\mu - j$, that is,

$$a_{\mu-j}(y,\delta\eta;\delta\lambda) = \delta^{\mu-j} a_{\mu-j}(y,\eta;\lambda) \quad \text{for } \delta > 0,$$

(II) $z^p a_{\mu-j}(y,\omega;1/z) = z^p a_{\mu-j}(y,\omega;\bar{z}/|z|^2)$ is smooth in all variables, where $|\omega| = 1$ and $z \in \overline{\Lambda}$, and is smooth down to $z = 0$.

We define the operator space $\Psi^{\mu,p}_\Lambda(Y, S_0)$ as those parameter-dependent operators $A(\lambda)$ having the following properties: If $\varphi \in C^\infty(Y)$ has support in a coordinate patch on Y, then $\varphi A(\lambda)\varphi$ is a pseudodifferential operator on $\mathbb{R}^{n-1}$ with a symbol in $S^{\mu,p}_{r,\mathrm{cl}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \Lambda)$. Here, $n = \dim M$. If $\varphi, \psi \in C^\infty(Y)$ have disjoint supports, then $\varphi A(\lambda)\psi \in \Psi^{-\infty,p}_\Lambda(Y, S_0)$ where this space is defined in (3.4). These spaces have the following composition rule [24]:

$$\Psi^{\mu,p}_\Lambda(Y,S_0) \circ \Psi^{\mu',p'}_\Lambda(Y,S_0) \subset \Psi^{\mu+\mu',p+p'}_\Lambda(Y,S_0). \tag{7.3}$$

We remark that

$$\bigcap_\mu \Psi^{\mu,p}_\Lambda(Y,S_0) \subset \Psi^{-\infty,p}_\Lambda(Y,S_0),$$

where $\Psi^{-\infty,p}_\Lambda(Y,S_0)$ is defined is (3.4), and the composition rule (7.3) continues to hold when either μ or μ' is $-\infty$.

In the following lemma, we put

$$A_\pm(\lambda) = \mathcal{C}_\pm P^{\mathrm{o}}_\pm(\lambda) + (\mathrm{Id} - \mathcal{C}_\pm)(\mathrm{Id} - P^{\mathrm{o}}_\pm(\lambda))$$

and

$$A^c_\pm(\lambda) = \mathcal{C}^c_\pm (P^c_\pm)^{\mathrm{o}}(\lambda) + (\mathrm{Id} - \mathcal{C}_\pm)(\mathrm{Id} - (P^c_\pm)^{\mathrm{o}}(\lambda)).$$

Theorem 7.1. *Outside a neighborhood of* $\lambda = 0$, *each of* $P_\pm(\lambda)$, $P^c_\pm(\lambda)$, $A_\pm(\lambda)$, $A_\pm(\lambda)^{-1}$, $A^c_\pm(\lambda)$, $A^c_\pm(\lambda)^{-1}$, $S_\pm(\lambda)^{-1}$, *and* $S^c_\pm(\lambda)^{-1}$ *is in* $\Psi^{0,0}_\Lambda(Y, S_0)$.

We remark that $P_\pm(\lambda)$ are only smooth for $\lambda \in \Lambda$ and not at $\lambda = 0$, while all the other operators are smooth at $\lambda = 0$, hence the phrase "outside a neighborhood of $\lambda = 0$" only refers to $P_\pm(\lambda)$.

Proof. We shall prove these statements only for $P_-(\lambda)$, $A_-(\lambda)$, and $S_-(\lambda)^{-1}$; the proofs for the other operators are analogous. To prove this for $P_-(\lambda)$, from (6.3) and (6.4), we see that

$$P_-(\lambda)\varphi = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{iy\cdot\eta}\, c(y, \eta; \lambda)\, \hat{\varphi}(\eta)\, d\eta,$$

where using the fact that $b(y, \eta)G = -Gb(y, \eta)$ (since this holds at the operator level), we have $c(y, \eta; \lambda) = c_0(y, \eta; \lambda) + \mathcal{O}\big((|\xi| + |\eta| + |\lambda|)^{-1}\big)$ with

$$c_0(y, \eta; \lambda) = -\frac{1}{2}\Big[G + \frac{Gb(y, \eta) + \lambda}{\sqrt{|\eta|^2 - \lambda^2}}\Big]G = \frac{1}{2}\Big[\mathrm{Id} - \frac{b(y, \eta) + \lambda G}{\sqrt{|\eta|^2 - \lambda^2}}\Big].$$

Moreover, going through the proof of Theorem 6.1, one can show that

$$c(y, \eta; \lambda) = c_0(y, \eta; \lambda) + c_1(y, \eta, \lambda), \quad c_1 \in S^{-1,-1}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \Lambda). \tag{7.4}$$

We claim that $c_0(y, \eta; \lambda)$ satisfies (I) and (II) above with $\mu = j = p = 0$. The fact that $c_0(y, \delta\eta; \delta\lambda) = c_0(y, \eta; \lambda)$ is clear. Set $\lambda = 1/z$ and $\eta = \omega$ with $|\omega| = 1$. We shall prove that $c_0(y, \omega; 1/z)$ is smooth at $z = 0$. Because the branch of the square root in the definition of c_0 is the negative real axis (see the proof of Theorem 6.1), one can check that

$$\sqrt{1 - 1/z^2} = \sqrt{-\lambda^2(1 - z^2)} = -i\lambda\sqrt{1 - z^2}.$$

Thus,

$$c_0(y, \omega; 1/z) = \frac{1}{2}\Big[\mathrm{Id} - \frac{b(y, \omega) + (1/z)G}{\sqrt{1 - 1/z^2}}\Big] = \frac{1}{2}\Big[\mathrm{Id} + \frac{z\, b(y, \omega) + G}{i\sqrt{1 - z^2}}\Big].$$

It follows that $c_0(y, \omega; 1/z)$ is smooth in all variables, including in z down to $z = 0$, and this formula implies that

$$c_0(y, \omega; 1/z)|_{z=0} = \tfrac{1}{2}[\mathrm{Id} - iG] = P^+. \tag{7.5}$$

Thus, $c_0 \in S^{0,0}_{r,\mathrm{cl}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \Lambda)$ (outside $\lambda = 0$; henceforth, we shall drop this phrase). One can also show that $c_1 \in S^{-1,-1}_{r,\mathrm{cl}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \Lambda)$. This shows that $P_-(\lambda) \in \Psi^{0,0}_{\Lambda}(Y, S_0)$, and hence by the composition rule (7.3),

$$P_-(\lambda) P^*_-(\lambda) + (\mathrm{Id} - P^*_-(\lambda))(\mathrm{Id} - P_-(\lambda)) \in \Psi^{0,0}_{\Lambda}(Y, S_0).$$

Moreover, by (7.4) the leading homogeneous symbol of this operator is

$$p(y, \eta; \lambda) = c_0(y, \eta; \lambda)\, c_0(y, \eta; \lambda)^* + (\mathrm{Id} - c_0(y, \eta; \lambda)^*)(\mathrm{Id} - c_0(y, \eta; \lambda)).$$

The properties of $c_0(y, \eta; \lambda)$ imply that $p(y, \eta; \lambda)$ satisfies (I) and (II) above with $\mu = j = p = 0$. We claim that $p(y, \eta; \lambda)^{-1}$ also has these properties. This symbol certainly satisfies (I), the only question is whether or not $p(y, \omega; 1/z)^{-1}$ is smooth at $z = 0$, for perhaps the invertibility of $p(y, \omega; 1/z)$ is destroyed at $z = 0$. However, (7.5) implies that

$$p(y, \omega; 1/z)|_{z=0} = P^+(P^+)^* + (\mathrm{Id} - P^+)^*(\mathrm{Id} - P^+) = P^+ + \mathrm{Id} - P^+ = \mathrm{Id}.$$

This shows that $p(y, \omega; 1/z)^{-1}$ is smooth at $z = 0$. Now this fact plus the usual parametrix construction, one can show that

$$(P_-(\lambda) P^*_-(\lambda) + (\mathrm{Id} - P^*_-(\lambda))(\mathrm{Id} - P_-(\lambda)))^{-1} \in \Psi^{0,0}_{\Lambda}(Y, S_0).$$

Composing this operator with $P_-(\lambda) P^*_-(\lambda)$ and using the composition rule (7.3) we obtain $P^{\mathrm{o}}_-(\lambda) \in \Psi^{0,0}_{\Lambda}(Y, S_0)$, and therefore (again by the composition rule (7.3)),

$$A_-(\lambda) = \mathcal{C}_- P^{\mathrm{o}}_-(\lambda) + (\mathrm{Id} - \mathcal{C}_-)(\mathrm{Id} - P^{\mathrm{o}}_-(\lambda)) \in \Psi^{0,0}_{\Lambda}(Y, S_0).$$

Another parametrix argument shows that $A_-(\lambda)^{-1} \in \Psi^{0,0}_{\Lambda}(Y, S_0)$, which implies that

$$S_-(\lambda)^{-1} := \; P^{\mathrm{o}}_-(\lambda) A_-(\lambda)^{-1} \mathcal{C}_- \in \Psi^{0,0}_{\Lambda}(Y, S_0).$$

Our theorem is now proved. □

Corollary 7.2. *Each of the differences* $A_{\pm}(\lambda)^{-1} - A^c_{\pm}(\lambda)^{-1}$, $S_+(\lambda)^{-1} - S^c_+(\lambda)^{-1}$, *and* $S_-(\lambda)^{-1} \mathcal{C}_- V(\mathrm{Id} - \mathcal{C}_+) - S^c_-(\lambda)^{-1}$, *is in* $\Psi^{-\infty,0}_{\Lambda}(Y, S_0)$.

Proof. By Proposition 3.2, we have $P_+(\lambda) - P_+^c(\lambda) \in \Psi_\Lambda^{-\infty,-2}(Y, S_0)$ and $\mathcal{C}_+ - \mathcal{C}_+^c \in \Psi_\Lambda^{-\infty,0}(Y, S_0)$, hence it follows from our composition rule (7.3) that

$$A_+^c(\lambda) - A_+(\lambda) = \mathcal{C}_+^c \circ (P_+^c)^{\mathrm{o}}(\lambda) + (\mathrm{Id} - \mathcal{C}_+^c) \circ (\mathrm{Id} - (P_+^c)^{\mathrm{o}}(\lambda)) - \mathcal{C}_+ \circ P_+^{\mathrm{o}}(\lambda) - (\mathrm{Id} - \mathcal{C}_+) \circ (\mathrm{Id} - P_+^{\mathrm{o}}(\lambda)) \in \Psi_\Lambda^{-\infty,0}(Y, S_0).$$

Since

$$A_+(\lambda)^{-1} - A_+^c(\lambda)^{-1} = A_+(\lambda)^{-1}(A_+^c(\lambda) - A_+(\lambda))A_+^c(\lambda)^{-1},$$

the composition rule (7.3) implies that $A_+(\lambda)^{-1} - A_+^c(\lambda)^{-1} \in \Psi_\Lambda^{-\infty,0}(Y, S_0)$. A similar proof works for the "−" operators.

The third assertion is proved like the second assertion, so we shall focus on the second one. By definition of $S_+(\lambda)^{-1}$ and $S_+^c(\lambda)^{-1}$, we can write

$$S_+(\lambda)^{-1} - S_+^c(\lambda)^{-1} = P_+(\lambda)A_+(\lambda)^{-1}\mathcal{C}_+ - P_+^c(\lambda)A_+^c(\lambda)^{-1}\mathcal{C}_+^c.$$

By Proposition 3.2, we have $P_+(\lambda) - P_+^c(\lambda) \in \Psi_\Lambda^{-\infty,-2}(Y, S_0)$ and $\mathcal{C}_+ - \mathcal{C}_+^c \in \Psi_\Lambda^{-\infty,0}(Y, S_0)$, and by our first assertion, $A_+(\lambda)^{-1} - A_+^c(\lambda)^{-1} \in \Psi_\Lambda^{-\infty,0}(Y, S_0)$, all of which together with the composition rule (7.3), imply that $S_+(\lambda)^{-1} - S_+^c(\lambda)^{-1} \in \Psi_\Lambda^{-\infty,0}(Y, S_0)$. □

Now by the definition of $K(\lambda)$ and $K^c(\lambda)$, Corollary 7.2 implies that $K(\lambda) - K^c(\lambda) \in \Psi_\Lambda^{-\infty,0}(Y, S_0)$. Therefore,

$$K(\lambda)K^c(\lambda)^{-1} = \mathrm{Id} + (K(\lambda) - K^c(\lambda))K^c(\lambda)^{-1} = \mathrm{Id} + \mathcal{S}(\lambda), \tag{7.6}$$

where

$$\mathcal{S}(\lambda) = (K(\lambda) - K^c(\lambda))K^c(\lambda)^{-1} \in \Psi_\Lambda^{-\infty,0}(Y, S_0),$$

since $K^c(\lambda)^{-1} = S_+^c(\lambda) + S_-^c(\lambda) \in \Psi_\Lambda^{0,0}(Y, S_0)$. Hence, by the expansion (3.4) and the fact that $K(\lambda)$ and $K^c(\lambda)$ are holomorphic over Λ, we have

$$\mathcal{S}(\lambda) \sim_{\Im\lambda\to\pm\infty} \sum_{k=0}^{\infty} \lambda^{-k}\mathcal{S}_k^{\pm}$$

for smoothing operators $\mathcal{S}_k^{\pm}$. Combining this expansion with Theorem 6.4, we get

Theorem 7.3. *For $\lambda \in \Lambda$, we have*

$$\det_F\,(K(\lambda)K^c(\lambda)^{-1}) \;\sim_{\Im\lambda\to\pm\infty}\; \sum_{k=0}^{\infty} a_k^{\pm}\,\lambda^{-k},$$

where $a_0^+ = 1$ and $a_0^- = \det_F\,U$.

8. The spectral invariants and $\det_F\,(K(\lambda)K^c(\lambda)^{-1})$

From now on we use the following notations:

$$\mathcal{D}_{\mathcal{C}} = \mathcal{D}_{\mathcal{C}_+} \sqcup \mathcal{D}_{\mathcal{C}_-}, \quad \mathcal{D}^c_{\mathcal{C}^c} = \mathcal{D}^c_{\mathcal{C}^c_+} \sqcup \mathcal{D}^c_{\mathcal{C}^c_-}.$$

In this section we relate the relative eta invariant

$$\eta(\mathcal{D}, \mathcal{D}_{\mathcal{C}}) = \eta(\mathcal{D}) - \eta(\mathcal{D}_{\mathcal{C}})$$

and relative ζ-determinant

$$\det{}_\zeta(\mathcal{D}^2 + v^2, \mathcal{D}^2_{\mathcal{C}} + v^2) = \frac{\det_\zeta(\mathcal{D}^2 + v^2)}{\det_\zeta(\mathcal{D}^2_{\mathcal{C}} + v^2)},$$

with similar relative invariant formulas holding for the operators on the cylinder N, to $\log\det_F\,(K(\lambda)K^c(\lambda)^{-1})$. The key result in this direction is

Proposition 8.1. *The following equalities hold for $\lambda \in \Lambda$:*

$$\begin{aligned}
&\partial_\lambda\big(\log\det_F\,(K(\lambda)K^c(\lambda)^{-1}) - \log\det_F\,(K(-\lambda)K^c(-\lambda)^{-1})\big)\\
&\quad = -2\mathrm{Tr}\left(\frac{\mathcal{D}}{\mathcal{D}^2-\lambda^2} - \frac{\mathcal{D}_{\mathcal{C}}}{\mathcal{D}^2_{\mathcal{C}}-\lambda^2} - \frac{\mathcal{D}^c}{(\mathcal{D}^c)^2-\lambda^2} + \frac{\mathcal{D}^c_{\mathcal{C}^c}}{(\mathcal{D}^c_{\mathcal{C}^c})^2-\lambda^2}\right),\\
&\partial_\lambda\big(\log\det_F\,(K(\lambda)K^c(\lambda)^{-1}) + \log\det_F\,(K(-\lambda)K^c(-\lambda)^{-1})\big)\\
&\quad = -2\mathrm{Tr}\left(\frac{\lambda}{\mathcal{D}^2-\lambda^2} - \frac{\lambda}{\mathcal{D}^2_{\mathcal{C}}-\lambda^2} - \frac{\lambda}{(\mathcal{D}^c)^2-\lambda^2} + \frac{\lambda}{(\mathcal{D}^c_{\mathcal{C}^c})^2-\lambda^2}\right).
\end{aligned}$$

Proof. The proofs of these formulas are similar, so we shall focus on the first. If $F(\lambda) = \log\det_F\,(K(\lambda)K^c(\lambda)^{-1})$, then from Theorem 4.4, we have

$$\begin{aligned}
&\partial_\lambda\big(F(\lambda) - F(-\lambda)\big) = \partial_\lambda F(\lambda) + (\partial_\lambda F)(-\lambda)\\
&\quad = -\mathrm{Tr}\big(\mathcal{D}(\lambda)^{-1} - \mathcal{D}_{\mathcal{C}}(\lambda)^{-1} - (\mathcal{D}^c(\lambda)^{-1} - \mathcal{D}^c_{\mathcal{C}^c}(\lambda)^{-1})\big)\\
&\qquad -\mathrm{Tr}\big(\mathcal{D}(-\lambda)^{-1} - \mathcal{D}_{\mathcal{C}}(-\lambda)^{-1} - (\mathcal{D}^c(-\lambda)^{-1} - \mathcal{D}^c_{\mathcal{C}^c}(-\lambda)^{-1})\big).
\end{aligned}$$

Since

$$\mathcal{D}(\lambda)^{-1} + \mathcal{D}(-\lambda)^{-1} = (\mathcal{D} - \lambda)^{-1} + (\mathcal{D} + \lambda)^{-1} = \frac{2\mathcal{D}}{\mathcal{D}^2 - \lambda^2}$$

with similar formulas holding for the resolvents of the other Dirac operators, we get our first equality. □

Since for any holomorphic branch of log around a point c, we have $\log(c + z) \sim \sum_{k=0}^{\infty} c_k z^k$ as $z \to 0$, by Theorem 7.3 it follows that $\log \det_F (K(\lambda)K^c(\lambda)^{-1})$ has expansions as $\Im\lambda \to \pm\infty$ that resemble the expansions in Theorem 7.3. Therefore, using the formulas in Proposition 8.1, we immediately obtain the following corollary.

Corollary 8.2. *As* $|\lambda| \to \infty$ *for* $\lambda \in \Lambda$, *we have*

$$\mathrm{Tr}\left(\frac{\mathcal{D}}{\mathcal{D}^2 - \lambda^2} - \frac{\mathcal{D}_{\mathcal{C}}}{\mathcal{D}_{\mathcal{C}}^2 - \lambda^2} - \frac{\mathcal{D}^c}{(\mathcal{D}^c)^2 - \lambda^2} + \frac{\mathcal{D}^c_{\mathcal{C}^c}}{(\mathcal{D}^c_{\mathcal{C}^c})^2 - \lambda^2}\right) = \mathcal{O}(\lambda^{-2}),$$

$$\mathrm{Tr}\left(\frac{1}{\mathcal{D}^2 - \lambda^2} - \frac{1}{\mathcal{D}_{\mathcal{C}}^2 - \lambda^2} - \frac{1}{(\mathcal{D}^c)^2 - \lambda^2} + \frac{1}{(\mathcal{D}^c_{\mathcal{C}^c})^2 - \lambda^2}\right) = \mathcal{O}(\lambda^{-3}).$$

In the following proposition, we write $\eta(\mathcal{D}, \mathcal{D}_{\mathcal{C}})$ in terms of resolvents.

Proposition 8.3. *We have*

$$\eta(\mathcal{D}, \mathcal{D}_{\mathcal{C}}) = \frac{2}{\pi}\int_0^{\infty} \mathrm{Tr}\left(\frac{\mathcal{D}}{\mathcal{D}^2 + v^2} - \frac{\mathcal{D}_{\mathcal{C}}}{\mathcal{D}_{\mathcal{C}}^2 + v^2} - \frac{\mathcal{D}^c}{(\mathcal{D}^c)^2 + v^2} + \frac{\mathcal{D}^c_{\mathcal{C}^c}}{(\mathcal{D}^c_{\mathcal{C}^c})^2 + v^2}\right) dv.$$

Proof. Consider the formula

$$\eta_A(s) = \mathrm{Tr}(A(A^2)^{-\frac{s+1}{2}}), \quad (A^2)^{-\frac{s+1}{2}} = \frac{1}{\pi i}\int_{\Gamma} \lambda^{-s}(A^2 - \lambda^2)^{-1}\, d\lambda$$

for the eta function of a self-adjoint operator A, where Γ is the contour $\Gamma = \{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda = \delta\}$ with $\delta > 0$ chosen so that the poles of $A(A^2 - \lambda^2)^{-1}$ lie on $[2\delta, \infty)$, and where λ^{-s} is defined via the standard branch. Now, a straightforward computation shows that $\mathcal{D}^c$ and $\mathcal{D}^c_{\mathcal{C}^c}$ have symmetric spectrum, hence $\eta_{\mathcal{D}^c}(s) = \eta_{\mathcal{D}^c_{\mathcal{C}^c}}(s) = 0$. Thus, with $\delta > 0$ chosen uniformly for $\mathcal{D}$, $\mathcal{D}_{\mathcal{C}}$, $\mathcal{D}^c$, and $\mathcal{D}^c_{\mathcal{C}^c}$, by Proposition 3.4 it follows that for $\operatorname{Re} s \gg 0$, we have

$$\begin{aligned}\eta_{\mathcal{D}}(s) - \eta_{\mathcal{D}_{\mathcal{C}}}(s) &= \eta_{\mathcal{D}}(s) - \eta_{\mathcal{D}_{\mathcal{C}}}(s) - \eta_{\mathcal{D}^c}(s) + \eta_{\mathcal{D}^c_{\mathcal{C}^c}}(s) \\ &= \frac{1}{\pi i}\int_\Gamma \lambda^{-s}\operatorname{Tr}\Big(\mathcal{D}(\mathcal{D}^2-\lambda^2)^{-1} - \mathcal{D}_{\mathcal{C}}(\mathcal{D}_{\mathcal{C}}^2-\lambda^2)^{-1} \\ &\qquad - \mathcal{D}^c((\mathcal{D}^c)^2-\lambda^2)^{-1} + \mathcal{D}^c_{\mathcal{C}^c}((\mathcal{D}^c_{\mathcal{C}^c})^2-\lambda^2)^{-1}\Big)\, d\lambda.\end{aligned}$$

According to the first estimate in Corollary 8.2, we can set $s = 0$ into this integral and conclude that

$$\begin{aligned}&\eta(\mathcal{D}) - \eta(\mathcal{D}_{\mathcal{C}}) \\ &\quad = \frac{1}{\pi i}\int_\Gamma \operatorname{Tr}\Big(\mathcal{D}(\mathcal{D}^2-\lambda^2)^{-1} - \mathcal{D}_{\mathcal{C}}(\mathcal{D}_{\mathcal{C}}^2-\lambda^2)^{-1} \\ &\qquad - \mathcal{D}^c((\mathcal{D}^c)^2-\lambda^2)^{-1} + \mathcal{D}^c_{\mathcal{C}^c}((\mathcal{D}^c_{\mathcal{C}^c})^2-\lambda^2)^{-1}\Big)\, d\lambda. \end{aligned} \tag{8.1}$$

Finally, because the poles of the resolvents lie on $[2\delta, \infty)$, we can shift the contour Γ to the imaginary axis $i\mathbb{R}$, and when we set $\lambda = iv$ in (8.1) we get our result. □

Proposition 8.4. *For $v \in \mathbb{R} \setminus \{0\}$, we have*

(1) $$\lim_{v\to\infty} \log \det_\zeta(\mathcal{D}^2 + v^2, \mathcal{D}^2_{\mathcal{C}} + v^2) - \log\det_\zeta((\mathcal{D}^c)^2 + v^2, (\mathcal{D}^c_{\mathcal{C}})^2 + v^2) = 0,$$

(2)

$$\begin{aligned}&\partial_v\big(\log \det_\zeta(\mathcal{D}^2 + v^2, \mathcal{D}^2_{\mathcal{C}} + v^2) - \log \det_\zeta((\mathcal{D}^c)^2 + v^2, (\mathcal{D}^c_{\mathcal{C}})^2 + v^2)\big) \\ &\quad = 2\operatorname{Tr}\left(\frac{v}{\mathcal{D}^2+v^2} - \frac{v}{\mathcal{D}^2_{\mathcal{C}}+v^2} - \frac{v}{(\mathcal{D}^c)^2+v^2} + \frac{v}{(\mathcal{D}^c_{\mathcal{C}^c})^2+v^2}\right).\end{aligned}$$

Proof. For $\operatorname{Re} s \gg 0$, by Proposition 3.4 it follows that

$$\begin{aligned}&\zeta_{\mathcal{D}^2+v^2}(s) - \zeta_{\mathcal{D}^2_{\mathcal{C}}+v^2}(s) - \zeta_{(\mathcal{D}^c)^2+v^2}(s) + \zeta_{(\mathcal{D}^c_{\mathcal{C}^c})^2+v^2}(s) \\ &\quad = \frac{1}{2\pi i}\int_\Gamma \lambda^{-s} G(v,\lambda)\, d\lambda, \end{aligned}\tag{8.2}$$

where

$$\begin{aligned}G(v,\lambda) &= \operatorname{Tr}\Big((\mathcal{D}^2+v^2-\lambda)^{-1} - (\mathcal{D}^2_{\mathcal{C}}+v^2-\lambda)^{-1} \\ &\qquad -((\mathcal{D}^c)^2+v^2-\lambda)^{-1} + ((\mathcal{D}^c_{\mathcal{C}^c})^2+v^2-\lambda)^{-1}\Big)\end{aligned}$$

and where $\Gamma = \{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda = \delta\}$ with δ any positive real number such that $0 < \delta < v^2$. (Note that the spectra of $\mathcal{D}^2 + v^2, \mathcal{D}^2_{\mathcal{C}} + v^2, (\mathcal{D}^c)^2 + v^2, (\mathcal{D}^c_{\mathcal{C}^c})^2 + v^2$ all lie in the interval $[v^2, \infty)$.) Here, λ^{-s} is defined via the standard branch. By the second estimate in Corollary 8.2, we have

$$G(v, \lambda) = \mathcal{O}\Big((\lambda - v^2)^{-3/2}\Big). \tag{8.3}$$

Taking the derivative of (8.2) with respect to s, multiplying the result by -1, and using the estimate (8.3) to justify setting $s = 0$, we obtain

$$\begin{aligned}&\log \det{}_\zeta(\mathcal{D}^2 + v^2, \mathcal{D}^2_{\mathcal{C}} + v^2) - \log \det{}_\zeta((\mathcal{D}^c)^2 + v^2, (\mathcal{D}^c_{\mathcal{C}})^2 + v^2)\\ &\quad = \frac{1}{2\pi i}\int_\Gamma \log \lambda\, G(v, \lambda)\, d\lambda.\end{aligned}$$

Taking $v \to \infty$ and using the estimate (8.3) implies (1). (2), we observe that $\partial_v G(v, \lambda) = -2v\, \partial_\lambda G(v, \lambda)$, which implies that

$$\begin{aligned}&\partial_v\big(\log \det{}_\zeta(\mathcal{D}^2 + v^2, \mathcal{D}^2_{\mathcal{C}} + v^2) - \log \det{}_\zeta((\mathcal{D}^c)^2 + v^2, (\mathcal{D}^c_{\mathcal{C}})^2 + v^2)\big)\\ &\quad = -\frac{v}{\pi i}\int_\Gamma \log \lambda\, \partial_\lambda G(v, \lambda)\, d\lambda\\ &\quad = \frac{v}{\pi i}\int_\Gamma \frac{1}{\lambda}\, G(v, \lambda)\, d\lambda,\end{aligned}$$

where we used the estimate (8.3) to integrate by parts. By Cauchy's formula, the right-hand side of this equation is exactly $2v\, G(v, 0)$, which is exactly the right-hand side of (2). $\square$

The following theorem, which follows from Propositions 8.1, 8.3, and 8.4, is the main result in this section.

Theorem 8.5. *We have*

(1)

$$\begin{aligned}&\eta(\mathcal{D}, \mathcal{D}_{\mathcal{C}})\\ &\quad = -\frac{1}{\pi i}\int_0^\infty \partial_v\big(\log \det{}_F\,(K(iv)K^c(iv)^{-1}) - \log \det{}_F\,(K(-iv)K^c(-iv)^{-1})\big)\, dv,\end{aligned}$$

(2) *and for* $v \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned}&\partial_v\big(\log \det{}_\zeta(\mathcal{D}^2 + v^2, \mathcal{D}^2_{\mathcal{C}} + v^2) - \log \det{}_\zeta((\mathcal{D}^c)^2 + v^2, (\mathcal{D}^c_{\mathcal{C}})^2 + v^2)\big)\\ &\quad = \partial_v\big(\log \det{}_F\,(K(iv)K^c(iv)^{-1}) + \log \det{}_F\,(K(-iv)K^c(-iv)^{-1})\big).\end{aligned}$$

We end this section with the following result that we will need shortly.

Proposition 8.6. *For positive* v *near* 0, *we have*

$$\det{}_\zeta(\mathcal{D}^2+v^2,\mathcal{D}^2_{\mathcal{C}}+v^2) = v^{2h_M}\cdot\det{}_\zeta(\mathcal{D}^2,\mathcal{D}^2_{\mathcal{C}})\big(1+o(1)\big),$$
$$\det{}_\zeta((\mathcal{D}^c)^2+v^2,(\mathcal{D}^c_{\mathcal{C}^c})^2+v^2) = \det{}_\zeta((\mathcal{D}^c)^2,(\mathcal{D}^c_{\mathcal{C}^c})^2)\big(1+o(1)\big).$$

Proof. If Π is the orthogonal projection onto the kernel of $\mathcal{D}$, then we can write

$$\operatorname{Tr} e^{-t(\mathcal{D}^2+v^2)} = e^{-tv^2}\operatorname{Tr} e^{-t\mathcal{D}^2} = e^{-tv^2}h_M + e^{-tv^2}\operatorname{Tr}\big(\Pi^\perp e^{-t\mathcal{D}^2}\big).$$

Hence,

$$\zeta_{\mathcal{D}^2+v^2}(s) = v^{-2s}\,h_M + \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-tv^2}\operatorname{Tr}(\Pi^\perp e^{-t\mathcal{D}^2})\,dt. \tag{8.4}$$

Since $\operatorname{Tr}\big(\Pi^\perp e^{-t\mathcal{D}^2}\big)$ vanishes exponentially as $t\to\infty$ and $e^{-tv^2}=1+\mathcal{O}(tv^2)$, the integral on the right-hand side in (8.4) equals $\zeta_{\mathcal{D}^2}(s)+\mathcal{O}(v^2)$. Taking the derivative with respect to s and multiplying by -1, we find that as $v\to 0^+$,

$$\log\det{}_\zeta(\mathcal{D}^2+v^2) = 2h_M\log v + \log\det{}_\zeta\mathcal{D}^2 + o(1).$$

A similar argument shows that $\log\det_\zeta(\mathcal{D}^2_{\mathcal{C}}+v^2) = \log\det_\zeta\mathcal{D}^2_{\mathcal{C}}+o(1)$ as $v\to 0^+$, since $\mathcal{D}_{\mathcal{C}}$ is invertible by definition of the Calderón projectors $\mathcal{C}_\pm$. This proves our proposition for $\det_\zeta(\mathcal{D}^2+v^2,\mathcal{D}^2_{\mathcal{C}}+v^2)$ as $v\to 0^+$. A similar proof gives our result for $\det_\zeta((\mathcal{D}^c)^2+v^2,(\mathcal{D}^c_{\mathcal{C}^c})^2+v^2)$ as $v\to 0^+$, since in this case both $\mathcal{D}^c$ and $\mathcal{D}^c_{\mathcal{C}^c}$ are invertible. □

9. Proof of the main theorems

For Theorem 1.1, we begin with formula (2) of Theorem 8.5, which implies that

$$\frac{\det_\zeta(\mathcal{D}^2+v^2,\mathcal{D}^2_{\mathcal{C}}+v^2)}{\det_\zeta((\mathcal{D}^c)^2+v^2,(\mathcal{D}^c_{\mathcal{C}})^2+v^2)} = C\det{}_F\,(K(iv)K^c(iv)^{-1})\cdot\det{}_F\,(K(-iv)K^c(-iv)^{-1}), \tag{9.1}$$

where C is a constant. To find the constant C, we take $v\to\infty$ on both sides and use (1) in Proposition 8.4 to see that the left-hand side of (9.1) tends to unity, and use

$$\lim_{v\to\infty}\det{}_F\,(K(iv)K^c(iv)^{-1}) = 1,\ \lim_{v\to\infty}\det{}_F\,(K(-iv)K^c(-iv)^{-1}) = \det{}_F\,U,$$

from Theorem 6.4, to get

$$C = \det_F (U^{-1}) = (-1)^{h_M} \det_F (\widehat{U}^{-1}),$$

where we used the definition of $\widehat{U}$. Substituting this expression into (9.1), we obtain

$$\frac{\det_\zeta(\mathcal{D}^2 + v^2, \mathcal{D}^2_{\mathcal{C}} + v^2)}{\det_\zeta((\mathcal{D}^c)^2 + v^2, (\mathcal{D}^c_{\mathcal{C}})^2 + v^2)} = (-1)^{h_M} \det_F (\widehat{U}^{-1}) \det_F (K(iv)K^c(iv)^{-1}) \cdot \det_F (K(-iv)K^c(-iv)^{-1}).$$

We can evaluate the right-hand side by Theorem 5.9: for positive v near 0, we have

$$\det_F (K(\pm iv)K^c(\pm iv)^{-1}) = (\pm 1)^{h_M} v^{h_M} (\det \mathcal{L})^{-1} \det_F \Big(\frac{\mathrm{Id} + \widehat{U}}{2}\Big)\big(1 + o(1)\big),$$

therefore as $v \to 0^+$,

$$\begin{aligned}\frac{\det_\zeta(\mathcal{D}^2 + v^2, \mathcal{D}^2_{\mathcal{C}} + v^2)}{\det_\zeta((\mathcal{D}^c)^2 + v^2, (\mathcal{D}^c_{\mathcal{C}})^2 + v^2)} &= v^{2h_M} (\det \mathcal{L})^{-2} \det_F (\widehat{U}^{-1}) \det_F \Big(\frac{\mathrm{Id} + \widehat{U}}{2}\Big)^2 \big(1 + o(1)\big)\\ &= v^{2h_M} (\det \mathcal{L})^{-2} \det_F \Big(\frac{2\mathrm{Id} + \widehat{U} + \widehat{U}^{-1}}{4}\Big)\big(1 + o(1)\big).\end{aligned}$$

On the other hand, by Proposition 8.6,

$$\frac{\det_\zeta(\mathcal{D}^2 + v^2, \mathcal{D}^2_{\mathcal{C}} + v^2)}{\det_\zeta((\mathcal{D}^c)^2 + v^2, (\mathcal{D}^c_{\mathcal{C}})^2 + v^2)} = v^{2h_M} \frac{\det_\zeta(\mathcal{D}^2, \mathcal{D}^2_{\mathcal{C}})\big(1 + o(1)\big)}{\det_\zeta((\mathcal{D}^c)^2, (\mathcal{D}^c_{\mathcal{C}^c})^2)\big(1 + o(1)\big)}.$$

Equating the previous two lines and then taking $v \to 0^+$, we conclude that

$$\det_\zeta(\mathcal{D}^2, \mathcal{D}^2_{\mathcal{C}}) = \det_\zeta((\mathcal{D}^c)^2, (\mathcal{D}^c_{\mathcal{C}^c})^2) (\det \mathcal{L})^{-2} \det_F \Big(\frac{2\mathrm{Id} + \widehat{U} + \widehat{U}^{-1}}{4}\Big).$$

The next lemma completes the proof of Theorem 1.1.

Lemma 9.1. *We have*

$$\det_\zeta((\mathcal{D}^c)^2, (\mathcal{D}^c_{\mathcal{C}^c})^2) = 2^{-\zeta_{D_Y^2}(0) - h_Y}.$$

Proof. By the main theorem of Loya and Park [28], we have

$$\det_\zeta(\mathcal{D}^c)^2 = 2^{\zeta_{D_Y^2}(0)+h_Y} e^{2C}, \quad \det_\zeta(\mathcal{D}^c_{\mathcal{C}^c})^2 = \left(2^{\zeta_{D_Y^2}(0)+h_Y} e^{C}\right)^2$$

where $C = (\Gamma(s)^{-1}\zeta_{D_Y^2}(s-1/2))'(0)$, and dividing completes the proof. □

Using Theorem 1.1 and the main result of Loya and Park [26], we prove the general formula in Theorem 1.2 for other boundary conditions $\mathcal{P}_1 \in \mathrm{Gr}^*_\infty(\mathcal{D}_-)$, $\mathcal{P}_2 \in \mathrm{Gr}^*_\infty(\mathcal{D}_+)$.

To state the result proved in [26], we first need to recall some notation from the introduction. Let $\kappa_1 : L^2(Y, S^+) \to L^2(Y, S^-)$ be the map that determines $\mathcal{P}_1$ as $\kappa_\pm$ does for $\mathcal{C}_\pm$. Let P_1 be the orthogonal projection of $L^2(Y, S_0)$ onto the finite-dimensional vector space $\mathrm{ran}(\mathcal{C}_-) \cap \mathrm{ran}(\mathrm{Id} - \mathcal{P}_1)$. Then we introduce a linear map $\mathcal{L}_1$ over $\mathrm{ran}(\mathcal{C}_-) \cap \mathrm{ran}(\mathrm{Id} - \mathcal{P}_1)$ defined by

$$\mathcal{L}_1 := -P_1 G \mathcal{R}_-^{-1} G P_1,$$

where $\mathcal{R}_-$ is the sum of the Dirichlet to Neumann maps on the double of M_-, that was introduced in [7]. In [26], we prove that $\mathcal{L}_1$ is a positive operator so that $\det \mathcal{L}_1$ is a positive real number. The main result of Loya and Park [26] is the following comparison theorem.

Theorem 9.2. *For the orthogonal projection* $\mathcal{P}_1 \in \mathrm{Gr}^*_\infty(\mathcal{D}_-)$, *the following comparison formulas hold*:

$$\frac{\det_\zeta \mathcal{D}^2_{\mathcal{P}_1}}{\det_\zeta \mathcal{D}^2_{\mathcal{C}_-}} = (\det\, \mathcal{L}_1)^2 \cdot \det_F \left(\frac{2\mathrm{Id} + \widehat{U}_1 + \widehat{U}_1^{-1}}{4}\right),$$

$$\tilde{\eta}(\mathcal{D}_{\mathcal{P}_1}) - \tilde{\eta}(\mathcal{D}_{\mathcal{C}_-}) = \frac{1}{2\pi i} \mathrm{Log} \det_F\, U_1 \mod \mathbb{Z},$$

where $\widehat{U}_1$ *is the restriction of* $U_1 := \kappa_- \kappa_1^{-1}$ *to the orthogonal complement of its* (-1)-*eigenspace.*

Similar results hold for the corresponding objects over M_+ with the proper changes taking care of the orientation. We shall use the notations κ_2, U_2, and $\mathcal{L}_2$ for the corresponding objects associated to the pair $(\mathcal{D}_+, \mathcal{P}_2)$. Theorem 1.2 in the introduction now follows easily: by Theorem 1.1, we have

$$\frac{\det_\zeta \mathcal{D}^2}{\det_\zeta \mathcal{D}^2_{\mathcal{C}_+} \cdot \det_\zeta \mathcal{D}^2_{\mathcal{C}_-}} = 2^{-\zeta_{D_Y^2}(0) - h_Y} (\det\, \mathcal{L})^{-2} \det_F \left(\frac{2\mathrm{Id} + \widehat{U} + \widehat{U}^{-1}}{4}\right)$$

and by Theorem 9.2, we have

$$\frac{\det_\zeta \mathcal{D}^2_{\mathcal{P}_1}}{\det_\zeta \mathcal{D}^2_{\mathcal{C}_-}} = (\det\, \mathcal{L}_1)^2 \cdot \det_F \left(\frac{2\mathrm{Id} + \widehat{U}_1 + \widehat{U}_1^{-1}}{4}\right),$$

$$\frac{\det_\zeta \mathcal{D}^2_{\mathcal{P}_2}}{\det_\zeta \mathcal{D}^2_{\mathcal{C}_+}} = (\det\, \mathcal{L}_2)^2 \cdot \det_F \left(\frac{2\mathrm{Id} + \widehat{U}_2 + \widehat{U}_2^{-1}}{4}\right),$$

then combining these equalities we get exactly the formula of Theorem 1.2.

Recall that if $f(t)$ is a smooth nonzero complex-valued function on an interval $[a, b]$, then the *winding number* $W(f) \in \mathbb{Z}$ of f is defined by the equality:

$$\log\, f(b) - \log\, f(a) = \mathrm{Log}\, f(b) - \mathrm{Log}\, f(a) + 2\pi i\, W(f), \tag{9.2}$$

where $\log f(t)$ is any continuous logarithm for $f(t)$ with $t \in [a, b]$ and Log denotes the principal value logarithm. An integral expression for the left-hand side is

$$\log\, f(b) - \log\, f(a) = \int_a^b \frac{f'(t)}{f(t)}\, dt. \tag{9.3}$$

We now prove Corollary 1.3 by following almost *verbatim* the proof for the ζ-determinant! By formula (1) of Theorem 8.5, we have

$$\eta(\mathcal{D}, \mathcal{D}_{\mathcal{C}}) = -\frac{1}{\pi i}\Big(\lim_{v\to\infty} \big(\log \det_F K(iv)K^c(iv)^{-1} - \log \det_F\, K(-iv)K^c(-iv)^{-1}\big)$$
$$- \lim_{v\to 0^+} \big(\log \det_F\, K(iv)K^c(iv)^{-1} - \log \det_F\, K(-iv)K^c(-iv)^{-1}\big)\Big).$$

By Theorem 6.4, we have

$$\lim_{v\to\infty} \det_F\, (K(iv)K^c(iv)^{-1}) = 1,\ \lim_{v\to\infty} \det_F\, (K(-iv)K^c(-iv)^{-1}) = \det_F\, U,$$

so by definition of the winding number (9.2), we have, modulo $2\pi i\mathbb{Z}$,

$$\lim_{v\to\infty} \big(\log \det_F\, K(iv)K^c(iv)^{-1} - \log \det_F\, K(-iv)K^c(-iv)^{-1}\big) \equiv -\mathrm{Log}\det_F\, U,$$

where the integer defect is just the winding number of $\det_F\, (K(\lambda)K^c(\lambda)^{-1})$ from $-i\infty$ to $i\infty$. We remark that via (9.3) this winding number can be explicitly computed by quadrature of $\mathrm{Tr}(k'(\lambda)k(\lambda)^{-1})$ with $k(\lambda) = K(\lambda)K^c(\lambda)^{-1}$ or in terms of the resolvents as shown in Theorem 4.4. Similarly, by Theorem 5.9, for positive v near 0, we have

$$\det_F\, (K(\pm iv)K^c(\pm iv)^{-1}) = (\pm 1)^{h_M}\, v^{h_M} (\det\, \mathcal{L})^{-1} \det_F \Big(\frac{\mathrm{Id} + \widehat{U}}{2}\Big)\big(1 + o(1)\big)$$

so by definition of the winding number (9.2), we have, modulo $2\pi i\mathbb{Z}$,

$$\lim_{v\to 0^+}\left(\log \det_F K(iv)K^c(iv)^{-1} - \log \det_F\ K(-iv)K^c(-iv)^{-1}\right) \equiv \begin{cases} 0, & h_M \text{ even} \\ -\pi i, & h_M \text{ odd}, \end{cases}$$

where the integer defect is just the winding number of $\det_F (K(\lambda)K^c(\lambda)^{-1})$ from $-iv$ to iv for $v > 0$ sufficiently small. These equalities imply that

$$\widetilde{\eta}(\mathcal{D}, \mathcal{D}_{\mathcal{C}}) = \frac{1}{2\pi i}\mathrm{Log}\,\det_F\ U \mod \mathbb{Z},$$

where the integer defect consists of the aforementioned winding numbers if h_M is even or is shifted by $(h_M - 1)/2$ if h_M is odd. This completes the proof of Corollary 1.3 when $\mathcal{P}_1 = \mathcal{C}_-$ and $\mathcal{P}_2 = \mathcal{C}_+$. The general case considered in Corollary 1.3 follows from applying the comparison Theorem 9.2 in a similar manner as we did for the ζ-determinant case and recalling that the integer defect for the eta formula in Theorem 9.2 is given in terms of winding numbers that originate from completely natural operators defined from $\mathcal{P}_1$ and $\mathcal{C}_-$ [26].

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